NETAJI SUBHAS OPEN UNIVERSITY

Choice Based Credit System
(CBCS)

SELF LEARNING MATERIAL

HPH
PHYSICS

Mathematical Methods in Physics- II

CC-PH-07

Under Graduate Degree Programme

PREFACE

In a bid to standardize higher education in the country, the University Grants Commission (UGC) has introduced Choice Based Credit System (CBCS) based on five types of courses *viz. core, discipline specefic general elective, ability and skill enhancement* for graduate students of all programmes at Honours level. This brings in the semester pattern, which finds efficacy in sync with credit system, credit transfer, comprehensive continuous assessments and a graded pattern of evaluation. The objective is to offer learners ample flexibility to choose from a wide gamut of courses, as also to provide them lateral mobility between various educational institutions in the country where they can carry their acquired credits. I am happy to note that the university has been recently accredited by National Assessment and Accreditation Council of India (NAAC) with grade ''A''.

UGC Open and Distance Learning (ODL) Regulations, 2017 have mandated compliance with CBCS for UGC 2020 programmes for all the HEIs in this mode. Welcoming this paradigm shift in higher education, Netaji Subhas Open University (NSOU) has resolved to adopt CBCS from the academic session 2021-22 at the Bachelors Degree Programme (BDP) level. The present syllabus, framed in the spirit of syllabi recommended by UGC, lays due stress on all aspects envisaged in the curricular framework of the apex body on higher education. It will be imparted to learners over the six semesters of the Programme.

Self Learning Materials (SLMs) are the mainstay of Student Support Services (SSS) of an Open University. From a logistic point of view, NSOU has embarked upon CBCS presently with SLMs in English / Bengali. Eventually, the English version SLMs will be translated into Bengali too, for the benefit of learners. As always, all of our teaching faculties contributed in this process. In addition to this we have also requisitioned the services of best academics in each domain in preparation of the new SLMs. I am sure they will be of commendable academic support. We look forward to proactive feedback from all stakeholders who will participate in the teaching-learning based on these study materials. It has been a very challenging task well executed, and I congratulate all concerned in the preparation of these SLMs.

I wish the venture a grand success.

Professor (Dr.) Subha Sankar Sarkar
Vice-Chancellor

Netaji Subhas Open University
Under Graduate Degree Programme
Choice Based Credit System (CBCS)
Subject : Honours in Physics (HPH)
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UG : Physics
(HPH)

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Unit–1 ❑ Fourier Series

Structure

1.0 Objectives

After reading this unit you will learn

1. Definitions of periodic and Orthogonal Functions.
2. Harmonic analysis, Fourier's theorem, Fourier Series, Fourier Co-efficients, Dirichlet's conditions
3. Some examples of applications of Fourier series in different branches of physics.

1.1 Introduction

Jean—Baptiste Joseph Fourier (21 March, 1768-16 May, 1830) was a French mathematician and Physcist, known as the investigator of Fourier series.

The idea of Fourier series encountered to develop in mathematical science and engineering. The applications of Fourier series include in the general heat equation, vibrational modes of structural elements in buildings, quantum harmonic oscillators etc.

1.2 Periodic function :

Any funciton which repeats itself regularly over a given interval, is called periodic function.

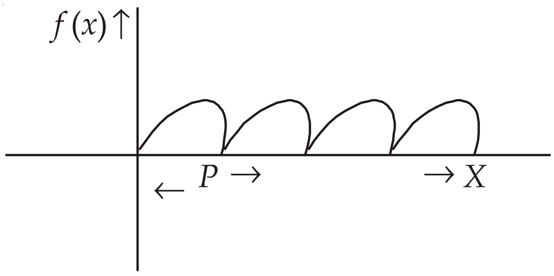


Fig. 1.1 An illustration of a periodic function over an interval *p*.

Here function $f(x)$ is periodic *i.e* $f(x + P) = f(x)$

1.3 Harmonic Analysis :

If a note has frequency *n*, then integer multiples of the frequency 2n, 3n and so on, are known as harmonics. The mathematical study of overlapping waves is called, harmonic analysis.

1.4 Orthogonal function :

The functions *f* and *g* are orthogonal when this integral of the product of functions over the interval

$$\int f^*(x)\,g(x)\,dx = 0$$

1.5 Fourier's theorem :

Fourier theorem is an expansion of a periodic function in terms of an infinite sum of sines and cosines. Mathematically, a periodic function $f(x)$ can be represented by the sum of harmonic terms as

$$f(x)=\frac{1}{2}a_0+c_1\cos x+a_2\cos 2x+......+a_n\cos n\ x$$

$$+\ b_1\ \sin\ x\ +\ b_1\ \sin\ 2x\ +\ ....+\ b_n\ \sin\ nx$$

$$=\ \frac{1}{2}a_0+\sum_{n=1}^{\alpha}\left(a_n\cos nx+b_n\sin nx\right)\quad ......(1.1)$$

It is a convergent series and a_0, a_n and b_n are called Fourier co-efficients.

1.6 Complex forms of Fourier series :

Substituting a_n = c_n cos θ_n and b_n = c_n sin θ_n in equation (i) we get,

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\alpha}\left(c_n\cos\theta_n\cos nx+c_n\sin\theta_n\sin nx\right)$$

$$=\frac{1}{2}a_0+\sum_{n=1}^{\alpha}c_n\cos\left(nx-\theta_n\right)\qquad ...\ (1.2)$$

where, $C_n=\sqrt{a_n^2+b_n^2}$ and $\theta_n=\tan^{-1}\frac{b_n}{a_n}$

Similarly, substituting cos $nx=\frac{e^{inx}+e^{-inx}}{2}$

$$sin\, nx=\frac{e^{inx}-e^{-inx}}{2i}$$

we get from equation (1.1)

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\alpha}\left(a_n\frac{e^{in\,x}+e^{-in\,x}}{2}+b_n\frac{e^{in\,x}-e^{-in\,x}}{2i}\right)$$

$$=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\frac{e^{in\,x}+e^{-in\,x}}{2}-ibn\frac{e^{inx}-e^{-inx}}{2}\right)$$

$$=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(\frac{a_n-ibn}{2}e^{inx}+\frac{a_n+ib_n}{2}e^{-inx}\right)$$

$$=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(\frac{a_n-ib_n}{2}e^{inx}\sum_{n=-1}^{-\infty}\frac{a_n+ib_n}{2}e^{inx}\right) \qquad ...(1.3)$$

Now, put $\frac{1}{2}a_0=C_0$

$$\left.\begin{aligned}C_n&=\frac{1}{2}(a_n-ib_n)\\ C_{-n}&=\frac{1}{2}(a_n-ib_n)\end{aligned}\right\}\text{ in equation (1.3)}$$

We get, $f(x)=C_0+\sum\limits_{n=1}^{\infty}C_ne^{in\,x}+\sum\limits_{n=-1}^{-\infty}C_{-n}e^{inx}$

$$=\sum_{n=-\infty}^{\infty}C_ne^{inx} \qquad ...\ (1.4)$$

Equation (1.4) represents the complex form of Fourier series and the co-efficient C_n is known as complex Fourier co-efficient.

Sine and Cosine integral for Fourier series in the integral $-\pi$ to π.

1. $\int_{-\pi}^{\pi} \sin nx\, dx = 0 \textit{ for all } n$

2. $\int_{-\pi}^{\pi} \cos nx\, dx = \begin{cases} 0 \text{ for } n \neq 0 \\ 2\pi \text{ for } n = 0 \end{cases}$

3. $\int_{-\pi}^{\pi} \sin^2 nx\, dx = \pi$

4. $\int_{-\pi}^{\pi} \cos^2 nx\, dx = \begin{cases} \pi \text{ for } n \neq 0 \\ 2\pi \text{ for } n = 0 \end{cases}$

5. $\int_{-\pi}^{\pi} \sin nx \sin mx\, dx = 0$ for $n \neq m$.

6. $\int_{-\pi}^{\pi} \cos nx \cos mx\, dx = 0$ for $n \neq m$.

7. $\int_{-\pi}^{\pi} \sin nx \cos mx\, dx = 0$ for all n and m.

The integral formulas of number (5), (6) and (7) for sines and cosines, are known as orthogonality relations.

1.7 Fourier co-efficients :

By multiplying equation (1.1) by cos $mx\, dx$ and integrating both sides in the interval $x = -\pi$ to π,

we obtain,

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \frac{1}{2}a_0\int_{-\pi}^{\pi}\cos mx\,dx + \int_{-\pi}^{\pi}\cos mx\left\{\sum_{n=1}^{\infty} a_n\cos mx\right\}dx$$

$$+\int_{-\pi}^{\pi}\cos mx\left\{\sum_{n=1}^{\infty} b_n\sin nx\right\}dx$$

For $m = 0$, as stated in the integral formulas (1-7), we can write,

$$\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2}a_0(2\pi)$$

or, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

For $m = n$ and m, $n > 0$,

$$a_n = \frac{1}{\pi}\cdot\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

By multiplying equation (1.1) by sin mx. dx and integrating bothsides in the interal $x = -\pi$ to π.

we obtain, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,$

where, $m = n$ and $m,n > 0$

$$\begin{cases} a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{cases}$$

1.8 Dirichlet's conditions :

(i) The function $f(x)$ is periodic, single valued and finite.

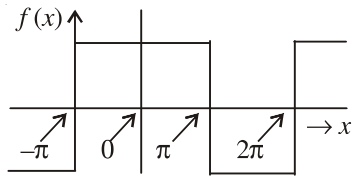


fig. (1.1a)

(ii) $f(x)$ is piece wise continuous that means there are finite number of discontinuities in any one period.

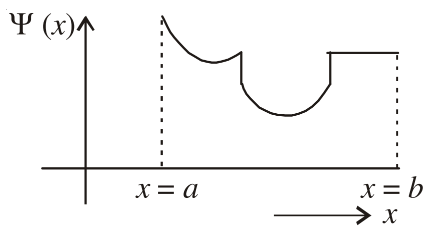


fig. (1.1b)

(iii) $f(x)$ has only a finite number of maxima and minima. The function $f(x) = \cos\left(\frac{1}{x}\right)$ has infinite numbers of maxima and minima.

So, Fourier expansion is not possible.

1.9 Even and Odd functions :

Cosine terms are even functions i.e., cos x = cos (– x) and sine terms are odd functions i.e., sin x = –sin (–x)

$$\text{Now, } f(x) = \underbrace{\frac{1}{2}\left[f(x)+f(-x)\right]}_{\text{Even part}} + \underbrace{\frac{1}{2}\left[f(x)-f(-x)\right]}_{\text{Odd part}}$$

In Fourier series cosine part represents even part and sine part represents odd part. Thus if $f(x)$ is even functions, then Fourier series having only cosine terms and if $f(x)$ is odd function, then only sine term will be their in Fourier series.

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{for } f(x) \text{ is even function}$$

and $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} b_n \sin nx$ for $f(x)$ is odd function.

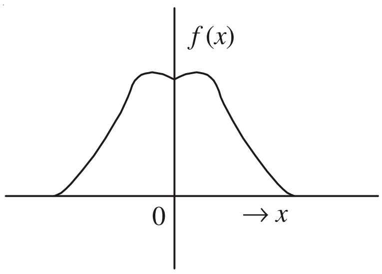


fig. 1.2 Even function

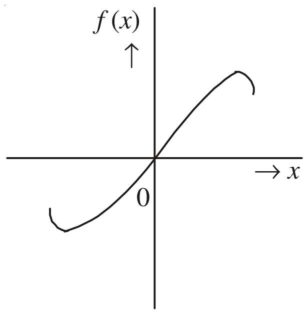


fig. 1.3 Odd function

For example, x^2, cos x .... are even function, where as sin x, x, tan x .... are odd function,

Graphically, an even function is symmetrical about y-axis and an odd function is symmetrical about the origin.

1.10 Applications of Fourier series :

Fourier series help us to slove the problems in the field of Fourier Analysis. By expanding the functions in terms of sines and cosines, one can easily manipulate functions that are discontinuous or difficult to represent analytically. The fields of electronics, quantum mechenics and eletro dynamics are enriched with the use of Fourier series :

1. Sawtooth wave : Mathematically it can be expressed as $f(x) = x$ for $-\pi \le x \le \pi$

Graphically it can be represented in the interval [– π, π], shown in fig. 1.4.

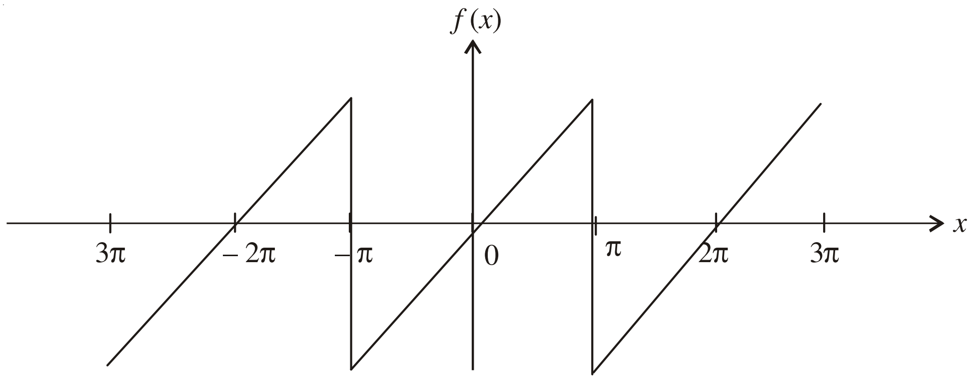


fig. 1.4

The Fourier series expansion for $f(x)$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$$

$f(x) = x$ is an odd function

Now, $\frac{a_0}{2}=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx=0$

Also, $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx=0$

since $f(x)\cos nx$ is odd function

and $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx$

$$=\frac{2}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\left[[\frac{x(-\cos nx)}{n}]_0^\pi - \int_0^\pi \frac{-\cos nx}{n} dx \right]$$

$$= \frac{2}{\pi n}\left\{ [-\pi \cos n\pi] + \left[\frac{\sin nx}{n}\right]_0^\pi \right\}$$

$$= -\frac{2}{n}(-1)^n$$

$$= \frac{2}{n}(-1)^{n+1}$$

The Fourier expansion is given by

$$f(x) = x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin nx}{n}$$

$$= 2[\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + ...]$$

2. Square Wave ;

$$f(x) = \begin{cases} 0, \text{for } -\pi \le x \le 0 \\ h, \text{for } -0 \le x \le \pi \end{cases}$$

The graphical representation is shown in fig. 1.5

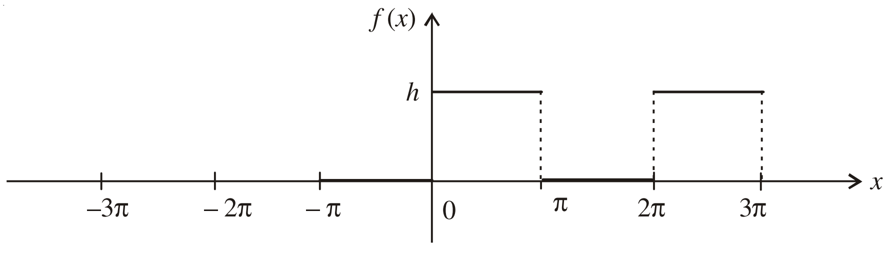


fig. 1.5

The Fourier series expansion is $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_x \cos nx+b_n \sin nx\right)$

Now, $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx+\int_{0}^{\pi} f(x)\,dx\right]$

$$= \frac{1}{\pi}\left[0+\int_{0}^{\pi} h\,dx\right]=h$$

Also, $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx=\frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$

$$=\frac{1}{\pi}\int_{0}^{\pi} h\cos nx\,dx=0$$

and $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} h\sin nx\,dx=\frac{h}{n\pi}\left(1-\cos n\pi\right)$

$$= \begin{cases} 0, & \text{for } n \text{ even} \\ \frac{2\mathrm{h}}{\mathrm{n}\pi} & \text{for } n \text{ odd} \end{cases}$$

$$f(x) = \frac{h}{2}+\frac{2h}{\pi}\sum_{n=1}^{\infty}\frac{\sin nx}{n}\left(n \text{ is odd only}\right)$$

$$=\frac{h}{2}+\frac{2h}{\pi}\left[\sin x+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\dots\dots\right]$$

The square wave in Fourier series representation containing only odd terms in the sine series. This means square wave contains high frequency.

3. Rectangular wave :

$$f(x)=\begin{cases} h, \text{for } 0 \le x \le \pi \\ -h, \text{for } \pi \le x \le 2\pi \end{cases}$$

The graphical representation is shown in fig. 1.6.

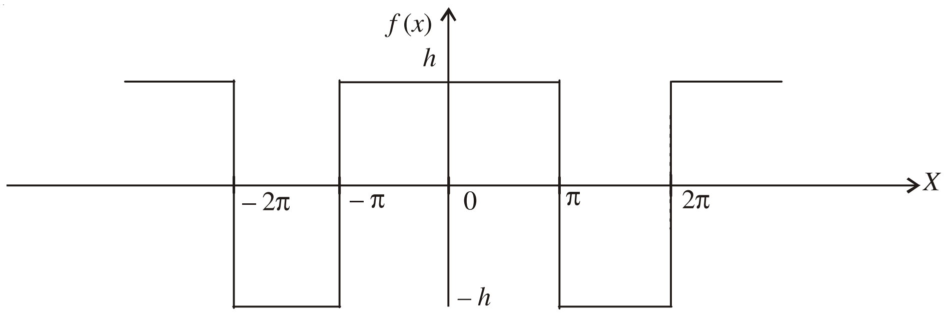


fig. 1.6

We have $f(x)=\dfrac{a_0}{2}+\sum\limits_{n=1}^{\infty}\left(a_n \cos nx+b_n \sin nx\right)$

Now, $a_0=\dfrac{1}{\pi}\int\limits_0^{2\pi} f(x)dx=\dfrac{1}{\pi}\left[\int\limits_0^{\pi} h\,dx-\int\limits_{\pi}^{2\pi} h\,dx\right]=0$

$$a_n=\frac{1}{\pi}\int\limits_{\pi}^{2\pi} f(x)\cos nx\,dx \quad =\frac{h}{\pi}\left[\int\limits_0^{\pi}\cos nx\,dx-\int\limits_{\pi}^{2\pi}\cos nx\,dx\right]=0$$

and $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

$$= \frac{h}{\pi}\left[\int_0^{\pi}\sin nx\,dx - \int_{\pi}^{2\pi}\sin nx\,dx\right]$$

$$= \frac{h}{n\pi}\left\{\left[-\cos nx\right]_0^{\pi} + \left[-\cos nx\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{h}{n\pi}\left\{(1-\cos n\pi)+(1-\cos n\pi)\right\}$$

$$= \frac{2h}{n\pi}(1-\cos n\pi)$$

$$= \begin{cases} 0, n \text{ is even} \\ \frac{4h}{n\pi}, n \text{ is odd} \end{cases}$$

Hence, $f(x) = \frac{4h}{\pi}\sum_{n=1}^{\infty}\frac{\sin nx}{n}$ (n is odd only)

$$= \frac{4h}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + ...\right]$$

4. Half-wave rectifier :

$f(x) = \sin x,\ 0 \le x < \pi$

$= 0,\ \pi \le x \le 2\pi$

The graphical representation of the function $f(x)$ is shown in fig. 1.7

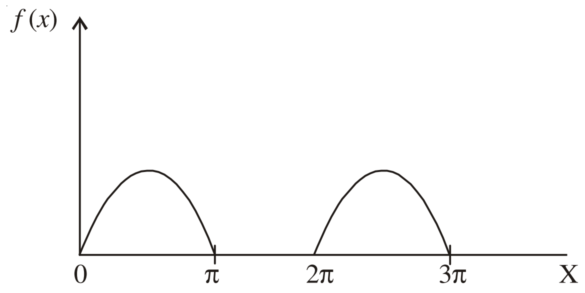


fig. 1.7

We have, $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

Now, $$a_0 = \frac{1}{\pi}\int_0^{\pi} \sin x \, dx + \int_{\pi}^{2\pi} 0.dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin x \, dx = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin x \cos nx \, dx$$

$$= \begin{cases} 0, \text{ for } n \text{ odd} \\ -\frac{2}{\pi} \frac{1}{n^2 - 1} \text{ for } n \text{ is even} \end{cases}$$

and $$b_n = \frac{1}{\pi}\int_0^{\pi} \sin x \sin nx \, dx = \begin{cases} 0 \text{ for all } n, \text{but } n \neq 1 \\ \frac{1}{2} \text{ for } n = 1 \end{cases}$$

The Fourier expansion of the function $f(x)$ is given by

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{n^2 - 1} \quad (n \text{ is even only})$$

5. Full-wave rectifier :

$f(x) = \sin x, \; 0 \le x < \pi$

$\quad = -\sin x, \; -\pi \le x \le 0$

The graphical representation of the function is shown in fig. 1.8.

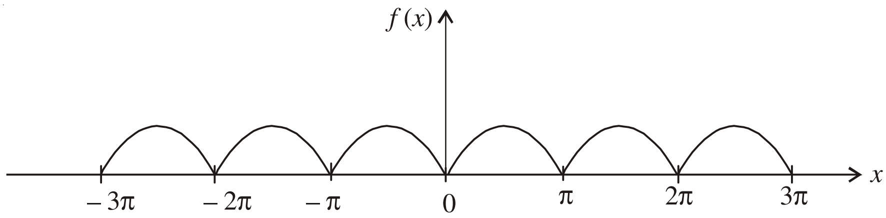


fig. 1.8

Here, $a_0 = \frac{1}{\pi}\int_{-\pi}^{0} f(x)dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\, dx$

$$= -\frac{1}{\pi}\int_{-\pi}^{0} \sin x\, dx + \frac{1}{\pi}\int_{0}^{\pi} \sin x\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} \sin x\, dx = \frac{4}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} -\sin x \cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} \sin x \cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \, dx$$

$$= \begin{cases} 0 \ \text{for } n\text{-odd} \\ -\frac{4}{\pi}\frac{1}{n^2-1}, \text{for } n\text{-even} \end{cases}$$

and $$b_n = \frac{2}{\pi}\int_0^{\pi} \sin x \sin nx \, dx = 0$$

Hence, Fourier expansion of the function $f(x)$ is given by

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{n^2-1} \quad (n \text{ is only even})$$

6. Riemann zeta function : The function is represented by,

$$\phi(t) = \sum_{n=1}^{\infty}\frac{1}{n^t} = \frac{1}{1^t} + \frac{1}{2^t} + \text{............}to \propto$$

Let, $f(x) = x^2, -\pi \le x \le \pi$

It can be graphically represented as in fig. 1.9.

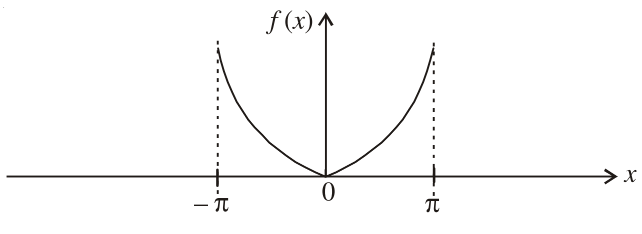


fig. 1.9

f(x) = x^2 is even function, so $b_n = 0$, Hence

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Now, $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx \, dx$

$$= \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx \, dx$$

$$= \frac{2}{\pi} \left[\frac{x^2}{n} \sin nx \right]_0^{\pi} - \frac{2}{\pi} \int_0^{\pi} \frac{2x}{n} \sin nx \, dx$$

$$= = -\frac{4}{\pi n} \int_0^{\pi} x \sin nx \, dx$$

$$= -\frac{4}{\pi n} \left[-\frac{x}{n} \cos x \right]_0^{\pi} - \frac{4}{n\pi} \int_0^{\pi} \frac{1}{n} \cos nx \, dx$$

$$= \frac{4}{n^2} \cos n\pi - \frac{4}{n\pi} \left[\frac{1}{n} \cdot \frac{1}{n} \sin nx \; dx \right]_0^{\pi}$$

$$= \frac{4}{n^2} (-1)^n$$

and $\quad a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{2\pi^2}{3}$

$$f(x) = x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\alpha} \frac{4}{n^2}(-1)^n \cos nx$$

Now put $x = \pi$, then

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\alpha} \frac{1}{n^2}(-1)^{2n}$$

or, $4\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2\pi^2}{3}$

or, $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

If we put $t = 2$, then

$$\phi(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

By following the same process, If we put $t = 4$.

$$\phi(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots\dots\dots\dots \infty$$

With the help of Fourier series expansion we get by putting $f(x) = x^4$, $-\pi \le x \le \pi$

$$\phi(4) = \frac{\pi^4}{90}$$

Hence, $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

1.11 Change of interval in Fourier expansion :

So far, the expansion was done in the interval [– π, π]. Now we discuss about the wider range. say [–*l*, *l*].

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\phi x + b_n \sin\phi x\right)$$

To calculate ϕ, Let us take $f(x) = f(x + 2l)$,

This is only true when $\phi = \frac{n\pi}{l}$

Hence, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right]$

If Dirichlet's condition is satisfied in the internal $-l \le x \le l$, then

co-efficient are

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx$$

$$b_n = \frac{1}{l}\int_{l}^{l} f(x)\sin\frac{n\pi x}{l}dx$$

and $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\, dx$

1.12 Half-range series :

For a function $f(x)$, ranging from $0 < x < l$ and $-l < x < 0$ is either odd or even. Then $f(x)$ will contain either cosine or sine series.

If $f(x)$ is to be expanded in a half-range cosine series in the interval $0 < x < l$, then we take $f(x) = f(-x)$ in the range $(-l < x < 0)$ i.e., $f(x)$ is even over the entire period $[-l, l]$.

Then $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$, where

$$a_0 = \frac{2}{l}\int_0^l f(x)dx, \text{ and}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

Now, if we want to expand in a half-range sine series in the internal $0 < x < l$, then we have to choose $f(x) = -f(-x)$ in the interval $-l < x < 0$. i.e, $f(x)$ is odd over the entire period $[-l, l]$.

Then $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$,

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{m\pi x}{l}dx$

Example 1. Exapand the Fourier series of the function $f(x) = x$ in the interval $-1 < x < 1$.

Solution : For the interval $-1 < x < 1$

$$f(x) = x = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\pi x + b_n \sin n\pi x\right)$$

Now, $a_0 = \int_{-1}^{1} f(x)dx = \int_{-1}^{1} x\, dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = 0$

$$a_n = \int_{-1}^{1} x \cos n\pi x\, dx = 0$$

and $b_n = \int_{-1}^{1} x \sin n\pi x\, dx = -\frac{2}{n\pi}(-1)^n$

Hence $f(x) = -\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n \frac{\sin n\pi x}{n}$

Example 2. Express the function $f(x) = x$ as a half-range sine series in the interval $0 < x < 2$.

Solution : We have, $f(x) = x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$, where

$$b_n = \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}dx$$

$$= \int_0^2 x \sin\frac{n\pi x}{2}dx$$

$$= \left[x\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}} - \frac{-\sin\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2} \right]_0^2$$

$$= -\frac{4}{n\pi}\cos\pi = -\frac{4}{n\pi}(-1)^n$$

$$= \begin{cases} \frac{4}{n\pi} \text{ for } n \text{ is odd} \\ -\frac{4}{n\pi} \text{ for } n \text{ is even} \end{cases}$$

Hence $f(x) = x = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{2}$

Example 3. Expand the function $f(x) = |x|$ *for* $-l < x < l$ and prove that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\dots = \frac{\pi^2}{8}$

Solution : We have, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right)$

Since $f(x) = |x|$ is even, so $b_n = 0$

Hence, $f(x) = \frac{a_0}{2} + \int_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}$

Now, $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{2}{l}\int_0^l x\,dx = l$

and $$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\int_{o}^{l} x\cos\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\left[x\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} - \frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right]_o^l$$

$$= \frac{2l}{n^2\pi^2}(\cos n\pi - 1)$$

$$= \frac{2l}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0, \ n \text{ is even} \\ -\frac{4l}{n^2\pi^2}, \ n \text{ is odd} \end{cases}$$

$$f(x) = |x| = \frac{l}{2} - \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l} \quad (n \text{ is odd only})$$

$$= \frac{l}{2} - \frac{4l}{\pi^2}\left[\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\frac{\cos 5\pi x}{l} + \ldots\ldots\ldots\right]$$

Put, $x = 0$, then

$$0=\frac{l}{2}-\frac{4l}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\ldots\ldots\right]$$

Hence $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\ldots\ldots\ldots\ldots=\frac{\pi^2}{8}$

Example 4. Expand $f(x) = x^2$ as Fourier series in the internal $-\pi < x < \pi$ and hence evaluate

(i) $\sum_{n=1}^{\infty}\frac{1}{n^2}$ and (ii) $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$

Solution : The function $f(x) = x^2$ is even in the interval $-\pi$ to π, hence $b_n = 0$

$$\therefore f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx$$

Now, $a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}x^2dx=\frac{\pi^2}{3}$

$$a_n=\frac{2}{\pi}\int_0^{\pi}x^2\cos nx\,dx=(-1)^n\frac{4}{n^2}$$

$$\therefore\quad f(x)=x^2=\frac{\pi^2}{3}-4\left[\frac{1}{1^2}\cos x-\frac{1}{2^2}\cos 2x+\frac{1}{3^2}\cos 3x\ldots\ldots\ldots\right]$$

(1) Putting $x = 0$ we get,

$$0=\frac{\pi^2}{3}-4\left(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}\ldots\ldots\ldots\ldots\ldots\right)$$

$$= \frac{\pi^2}{3} - 4\left[\left\{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ...\right\} - \frac{2}{2^2} - \frac{2}{4^2}.........\right]$$

$$= \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{1}{n^2} + 4\left(\frac{2}{2^2}\right)\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$= \frac{\pi^2}{3} - 2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

or, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

(ii) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...$

$$= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}................ + \frac{1}{2^2} + \frac{1}{4^2} + ... - \frac{1}{2^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$= \sum_{n=1}^{\infty}\frac{1}{n^2} - \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$= \frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$= \frac{3}{4}\cdot\frac{\pi^2}{6}$$

$$= \frac{\pi^2}{8}$$

Example : 5. An alternating current after passing through a rectification has the form

$$i = i_0 \sin x \qquad \text{for } 0 \le x \le \pi$$
$$= 0 \qquad \text{for } \pi \le x \le 2\pi$$

Where i_0 is the maximum current and the period in 2π. Express i as a Fourier series.

Solution : $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Now, $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} i_0 \sin x\, dx = \frac{2i_0}{\pi}$$

$$a_n = \frac{i_0}{\pi}\int_0^{\pi} \sin x \cos nx\, dx$$

$$= \frac{i_0}{2\pi}\int_0^{\pi}\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{i_0}{2\pi}\left[-\frac{\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{(n-1)}\right]_o^{\pi}$$

$$= \frac{i_0}{2\pi}\left[-\frac{1-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi - 1}{n-1}\right]$$

$$\therefore\ a_1 = 0,\ a_2 = -\frac{2i_0}{3\pi},\ a_3 = 0,\ a_4 = -\frac{2i_o}{15\pi},$$

$$a_5 = 0,\ a_0 = -\frac{2i_0}{35\pi},\ \ldots\ldots\ldots\ldots$$

$$b_n = \frac{i_0}{\pi}\int_0^{\pi} \sin x \sin nx\, dx$$

$$= \frac{i_0}{2\pi}\int_0^{\pi}\left[\cos(n-1)x - \cos(n+1)x\right]dx$$

$$= \frac{i_o}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi}$$

$$b_1 = \frac{i_o}{\pi}\int_0^{\pi}\sin^2 x\,dx = \frac{i_0}{2\pi}\int_0^{\pi}(1-\cos 2x)\,dx = \frac{i_0}{2}$$

$$\therefore \quad f(x) = \frac{i_0}{\pi} + \frac{i_0}{2}\sin x - \frac{2i_0}{3\pi}\cos 2x - \frac{2i_0}{15\pi}\cos 4x - \frac{2i_0}{35\pi}\cos 6x \ldots\ldots\ldots\ldots.$$

$$= \frac{i_0}{\pi} + \frac{i_0}{2}\sin x - \frac{2i_0}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1}$$

1.13 Summary

1. Complex forms of Fourier series and even and odd functions and change of interval in Fourier expansion have been discussed.
2. Various types of wave form are discussed with reference to the example of electronics.

1.14 Review Questions and Answer :

1. Explain the periodic functon and state the Fourier's theorem.

Ans. See articles no. (1.2) and (1.5)

2. Describe the complex form of Fourier's theorem.

Ans. See article no. (1.6)

3. What is odd and even function? Explain with example.

Ans. See article no. (1.9)

4. (a) What are the Dirichlet's conditions in Fourier's series? (b) What is the limitations of Fourier's theorem

Ans. (a) See article no. (1.8).

(b) We already discuss the Dirichlet's condition in Fourier's series. Therefore these conditions impose the limitations in the application of Fourier's theorem.

5. If F(x) have a Fourier Series expansion

$$\mathbf{F(x)} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx, \text{ \textbf{then}}$$

$$\textbf{prove that } \frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2\, dx = \frac{1}{\pi}a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

Ans. We have,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx\right]^2 dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{4}a_0^2 dx + \frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty} a_n^2 \cos^2 nx.dx + \frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty} b_n^2 \sin^2 nx\, dx$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty} a_0 a_n \cos nx\, dx + \frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty} a_0 b_n \sin nx\, dx + \frac{2}{\pi}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty} a_n \cos nx \sum_{n=1}^{\infty} b_n \sin nx\, dx$$

$$= \frac{1}{4\pi}2a_0^2\pi + \frac{1}{\pi}\left[\sum_{n=1}^{\infty}\pi a_n^2 + \sum_{n=1}^{\infty}\pi b_n^2\right] + 0 + 0 + 0 = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

$$\text{Hence, } \frac{1}{\pi}\int_{-\pi}^{\pi}\left[F(x)\right]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

The above relation is called Parseval relation.

6. Show that, if a real function $f(x)$ be expanded into complex Fourier series, that is $f(x)=\sum_{-\infty}^{\infty} a_n e^{inx}$, then show that C_{-n} is the complex conjugate of c_n.

Ans. We have, $a_n = \frac{1}{2n}\int_{-\pi}^{\pi} f(x)e^{inx}dx$

$$\therefore \; C_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = c_n^* \text{ where } c_n^* \text{ is the complex conjugate of } c_n.$$

7. Expand the function $f(x) = e^{bx}$ in the interval $[-\pi, \pi]$ in Fourier series where b is non-zero constant, and show that

(i) $$\frac{\pi \cos h\, bx}{2 \sin h\, b\pi} = \frac{1}{2b} + \sum_{n=1}^{\infty} (-1)^n \frac{b}{b^2+n^2} \cos nx \quad (-\pi \le x \le \pi)$$

(ii) $$\frac{\pi \cos h\; bx}{2 \sin h\; b\pi} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{b^2+n^2} \sin nx \; (-\pi \le x \le \pi)$$

Ans. We have the function $f(x) = e^{bx}$

Now, co-efficients of expansion are

$$\therefore a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{bx}dx = \frac{e^{b\pi} - e^{-b\pi}}{b\pi} = \frac{2 \sin h\, b\pi}{b\pi}$$

$$\therefore \; a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{bx}\cos nx . dx = \frac{1}{\pi}\left[\frac{b\cos nx + n \sin nx}{b^2+n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{(-1)^n}{\pi} \cdot \frac{2b}{b^2+n^2} \sin \text{h}.b\pi, \text{ and}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{b^x} \sin nx \, dx = \frac{1}{\pi} \left[\frac{b \sin bx - n \cos nx}{b^2+n^2} \right]_{-\pi}^{\pi}$$

$$= \frac{(-1)^{n-1}}{\pi} \frac{2n}{b^2+n^2} \sin h \, b\pi$$

Hence, we can write $f(x) = e^{bx}$ in Fourier expansion form

$$f(x) = e^{bx} = \frac{\sin h \, b\pi}{b\pi} + \sum_{n=1}^{\infty} \frac{(-1)^n}{\pi} \frac{2b}{b^2+n^2} \sin h \, b\pi \cos nx$$

$$+ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\pi} \frac{2n}{b^2+n^2} \sin h \, b\pi \sin nx$$

$$= \frac{2 \sin h \, b\pi}{\pi} \left[\frac{1}{2b} + \sum_{n=1}^{\infty} \frac{(-1)^n}{b^2+n^2} (b \cos nx - n \sin nx) \right] \quad \text{... (1)}$$

Replacing x by (–x) in equation (ii) we get,

$$e^{-bx} = \frac{2 \sin \text{h} \, b\pi}{\pi} \left[\frac{1}{2b} + \sum_{n=1}^{\infty} \frac{(-1)^n}{b^2+n^2} (b \cos nx + n \sin nx) \right] \quad \text{... (2)}$$

Now, adding equation (1) and equation (2) we get,

$$e^{bx}+e^{-bx}=\frac{2\sin h\, b\pi}{\pi}\left[\frac{1}{b}+\sum_{n=1}^{\infty}\frac{2b(-1)^n}{b^2+n^2}\cos nx\right]$$

$$\text{or,}\quad \frac{2\pi\cos h\; bx}{2\sin h\; b\pi}=\frac{1}{b}+\sum_{n=1}^{\infty}\frac{2b(-1)^n}{b^2+n^2}\cos nx$$

$$\therefore\quad \frac{\pi\cos h bx}{2\sin h b\pi}=\frac{1}{2b}+\sum_{n=1}^{\infty}(-1)^n\frac{b}{b^2+n^2}\cos nx \quad \textbf{(Proved)}$$

Now subtracting equation (1) and (2) we get,

$$e^{bx}-e^{-bx}=\frac{2\sin h\; b\pi}{\pi}\left[-\sum_{n=1}^{\infty}(-1)^n\frac{2n\sin nx}{b^2+n^2}\right]$$

$$\text{or,}\quad \frac{2\pi\sin h\; bx}{2\sin h\; b\pi}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n\sin nx}{b^2+n^2}$$

$$\therefore\quad \frac{\pi\sin \mathrm{h}\, bx}{2\sin \mathrm{h}\, b\pi}=\sum_{n=1}^{\alpha}(-1)^{n-1}\frac{n}{b^2+n^2}\sin nx \quad \textbf{(Proved)}$$

8. If $f(x)=x(0\le x\le\pi/2)\;=\;\pi-x\left(\frac{\pi}{2}\le x\le\pi\right)$, the express the function as a sine and cosine series and show that $1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\ldots\ldots\ldots\ldots=\frac{\pi^2}{8}$

Ans. The graphical representation of the function is shown below :

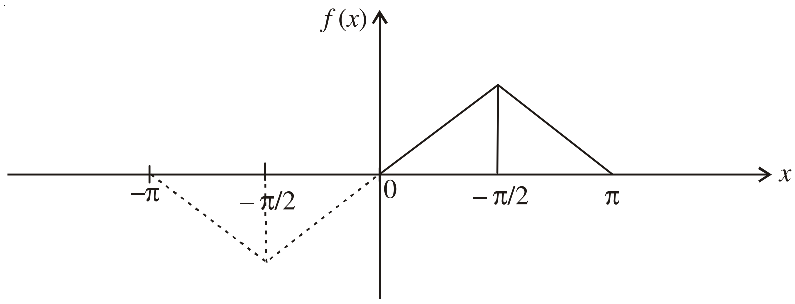


fig. : 1.10

The function is defined only over half range (0 to π), hence it can be represented by either cosine or sine series. When we represented as cosine series the function repeated in the range (–π to 0) as shown in dotted line i.e., as an even function and is given by $f(x) = f(-x)$.

The function given in the range (0 to π) is odd i.e., $f(x) = -f(-x)$ and can be represented by sine series.

For cosine series, the expansion over the half range (0 ro π) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \text{ where}$$

$$a_0 = \frac{1}{\frac{\pi}{2}} \int_0^{\pi} f(x) dx$$

$$\therefore \quad a_0 = \frac{2}{\pi}\left[\int_0^{\pi/2} x \; dx + \int_0^{\pi} (\pi - x) \; dx\right] = \frac{2}{\pi} \cdot \frac{\pi^2}{4} = \frac{\pi}{2}$$

and $a_n = \frac{1}{\frac{\pi}{2}} \int_0^{\pi} f(x) \cos nx \; dx$

Unit–2 ❑ Frobenius Method and Special Function

Structure :

2.0 Objectives

1. To know what is Frobenius method.
2. To know Legendre differential equation, Bessel's equation, Hermite's equation and their generating functions, recurrence formula and orthogonalities.

2.1 Introduction

As we have already remarked, in some simple cases it is possible to find the general solution of the homogeneous equation

$y'' + Py' + Qy = 0$ Equation (A)

in terms of familiar elementary functions. For the most part, however, the equations of this type having the greatest significance in theoretical physics are beyond the reach or elementary methods, and one has no choice but to resort to the method of series solution.

The central fact about the homogeneous equation (A) is that the behaviour of its solutions near a point x_0 depends on the behaviour of the coefficients P and Q near this point. A point x_0 is called an **ordinary point** of Eqn. (A) if both P and Q are analytic at x_0, in the sense that P(z) and Qtz), looked upon as functions of the complex variable z, are analytic at $z = x_0$. It is well known to us that in this case both P and Q will have Taylor series expansions in some neighbourhoods of the point x_0. Any point that is not an ordinary point of Eqn. (A) is called a **singular point** of Eqn. (A).

A singular point x_0 of Eqn. (A) is said to be **regular** if both $(x - x_0)$ P and $(x - x_0)^2Q$ are analytic at x_0, and **irregular** otherwise.

A singular point x_0 of Eqn. (A) is said to be **regular** if both $(x - x_0)P$ and $(x - x_0)^2Q$ are analytic at x_0, and irregular otherwise.

The reason behind such a classification of points in relation to a given homogeneous linear differential equation will become clear as we proceed.

2.2 Frobenius Method :

This method is developed by the German mathematician Ferdinand Georg Frobenius (1849-1917) to find out infinite series solution for a second-order ordinary differential equation of the form

$$P(x)\frac{d^2u}{dx^2}+Q(x)\frac{du}{dx}+R(x)u=0 \qquad \text{... (2.2.1)}$$

$$\therefore \quad \frac{d^2u}{dx^2}+A(x)\frac{du}{dx}+B(x)u=a \qquad \text{... (2.2.2), where}$$

$$A(x)=\frac{Q(x)}{P(x)} \text{ and } B(x)=\frac{R(x)}{P(x)} \text{ are the functions of } x.$$

The solutions of the differential equation

$$\frac{d^2u}{dx^2}-u=0 \text{ is } u = Ae^{x} + Be^{-x}$$

$$\text{and } \frac{d^2u}{dx^2}+u=0 \text{ is } u = A \sin x + B \cos x$$

The above two equations are valid for power series solution

Let us take the differential equation

$$\frac{d^2u}{dx^2}+\frac{1}{x^4}u=0,$$ having the solutions

$u = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots$

On solving the equation, we obtain $a_0 = a_1 = a_2 = \ldots\ldots\ldots\ldots 0$. So there is no series satisfying the above differential equation. Before going to the discussion of existance or non-existance of series solution of the form of equation. (2.2.1) we must have to know about the ordinary or singular points of the differential equation.

2.2.1 Ordinary and singular point and analytic function :

If $P(a)\neq 0$, then A(x) and $B(x)$ have finite values and $x = a$ is called an ordinary point of the equation. For ordinary point power series method exists.

If $P(a)=0$, then $x = a$ is called singular point and at that point $A(x)$ and $B(x)$ have the infinite values, At $x = a$,if $(x - a)\ A\ (x)$ and $(x - b)^2\ B(x)$ have finite values, then solution can be developed by power series method.

A function is said to be analytic or regular at a point $x = a$, if the function is single valued, continues derivatives and expand as Taylor expansion at $x = a$ i. e.

$$\sum\frac{f^n(a)}{n!}(x-a)^n \text{ exists.}$$

2.2.2 Existence of power series method :

The theorem by which we can examine whether the power series solution exists or not is called **Frobenius-Fuch's theorem.** The theorem is discussed below :

(i) If $x = a$ is ordinary point of eqation (2.2.1), then solution can be expressed as

$$u(x)=a_0+a_1(x-a)+a_3(x-a)^2 \quad \ldots. \ (2.2.3)$$

(ii) If $x = a$ is a regular or non-essential singular point, then solution can be expressed by $u(x)=(x-a)^m\left[a_0+a_1(x-a)+a_2(x-a)^2+\ldots..\right]$ ... (2.2.4)

The equations (2.2.3) and (2.2.4) are convergent at every print within the circle, drawn by taking the radius of the circle $x = a$.

Some examples are illustrated below :

Example 1. $\dfrac{d^2u}{dx^2}+x\dfrac{du}{dx}+x^2u=0$

By compairing with equation (2.2.1) we obtain $P(0)\neq 0$, is an ordinary point. Hence series solution method is eligible.

Example 2. $x^2\dfrac{d^2u}{dx^2}+x\dfrac{du}{dx}+u=0$

By compaining with equation (2.2.1) we obtain $P(0) = 0$ i.e., $x = 0$ is singular point.

The equation can be written as $\dfrac{d^2u}{dx^2}+\dfrac{1}{x}\dfrac{du}{dx}+\dfrac{1}{x^2}u=0$,

Hence, $(x-0)\cdot\dfrac{1}{x}$ and $(x-0)^2\cdot\dfrac{1}{x^2}$ have the finite values at $x = 0$, so the singular point is non-essential or regular and solution can be expressed by power series method.

Example 3. Laguerre equation is given by

$$x\frac{d^2u}{dx^2}+(1-x)\frac{du}{dx}+nu=0$$

or, $$\frac{d^2u}{dx^2}+\left(\frac{1}{x}-1\right)\frac{du}{dx}+\frac{n}{x}u=0$$

Hence $(x-0)\left(\dfrac{1-x}{x}\right)$ and $(x-0)^2\cdot\dfrac{n}{x}$ hvae the finite values at $x = 0$ and $x = 0$ is non-essential or regular point and can be expressed by power series method.

Example 4. $\dfrac{d^2u}{dx^2}+\dfrac{1}{x^4}u=0$

at $x = 0$, $(x-0)^2 \cdot \frac{1}{x^4} \to \infty$. Hence the singular point is essential and series solution method is not applicable

Example 5. $x^4 \frac{d^2u}{dx^2} + x^3 \frac{du}{dx} + xu = 0$

or, $\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} + \frac{1}{x^3}u = 0$

For above equation $(x-0)\frac{1}{x}|_{x=0} = 1$

but, $(x-0)^2 \cdot \frac{1}{x^3}|_{x=0} \to \infty$. Hence the singular point is essential and series solution method is not applicatble.

2.2.3. Steps for solving series solution :

(i) First we have to check the ordinary and singular point.

(ii) Then check equation (2.2.3) or eqation (2.2.4), which one be the solution of the equation.

(iii) Put $u, \frac{du}{dx}$ and $\frac{d^2u}{dx^2}$ in said equation.

(iv) Find the co-efficients a_0, a_1, a_2,..........

(v) Finally put the values in equation (2.2.3) or equation (2.2.4) and get the final result.

2.3. Legendre differential equation :

Legendre differential equation is given by

$$\left(1-x^2\right)\frac{d^2u}{dx^2}-2x\frac{du}{dx}+n(n+1)u=0 \quad \text{... (2.3.1)}$$

Here n is real constant

$$\text{or, } \frac{d^2u}{dx^2}-\frac{2x}{1-x^2}\cdot\frac{du}{dx}+\frac{n(n+1)}{1-x^2}u=0 \quad \text{... (2.3.2)}$$

Comparing equation (2.2.1) we obtain at $x=\pm1,\ P(\pm1)=0$.

Hence $x=\pm1$ is called singular point and

$$A(x)|_{x=1}=(x-1)\frac{-2x}{1-x^2} \text{ is finite and } A(x)\Big|_{x=-1}=(x+1)\frac{-2x}{\left(1-x^2\right)} \text{ is also finite}$$

Again, $B(x)\Big|_{x=1}=(x-1)^2\dfrac{n(n+1)}{\left(1-x^2\right)}$ is finite

and $B(x)\Big|_{x=-1}=(x+1)^2\dfrac{n(n+1)}{\left(1-x^2\right)}$ is also finite.

By the above argument, we must say $x=\pm1$ are non-essential or regular singular points.

Now, at $x = 0$, $P(0)=1-x^2|_{x=0}=1$ i.e., $P(0)\neq 0$.

Hence $x = 0$ is ordinary point and the trial solution can be expressed as equation (2.2.3) i.e.

$$u = \sum_{m=0}^{\infty} c_m x^m \qquad \text{... (2.3.3)}$$

Putting the value of equation (2.3.3) in equation (2.3.1) we get

$$\left(1-x^2\right)\sum_{m=0}^{\infty} m(m-1)c_m x^{m-2} - 2x\sum_{m=0}^{\infty} c_m \; m x^{m-1} + n(n+1)\sum_{m=0}^{\infty} c_m x^m = 0$$

or, $\displaystyle\sum_{m=0}^{\infty} m(m-1)c_m x^{m-2} - \sum_{m=0}^{\infty} m(m-1)c_m x^m - 2\sum_{m=0}^{\infty} c_m m x^{m+1} + n(n+1)\sum_{m=0}^{\infty} c_m x^m = 0$

To get the same power of x, we can write the above equation in the form

To get the same power x^5, we put $m = s+2$

Hence, $\displaystyle\sum_{s=0}^{\infty}(s+2)(s+1)c_{s+2}x^5 - \sum_{s=2}^{\infty} s(s-1)c_s x^5$

$$-\sum_{s=1}^{\infty} 2sc_s x^5 + \sum_{s=0}^{\infty} n(n+1)c_s x^5 = 0 \qquad \text{...}\rightarrow \text{(2.3.4)}$$

From equation (2.3.4) we get,

$$\boxed{C_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)}C_s} \qquad \text{...}\rightarrow \text{(2.3.5)}$$

This is called recurrence relation or recurssion formula.

$$C_2 = -\frac{n(n+1)}{2!}C_0 \qquad\qquad C_3 = -\frac{(n-1)(n+2)}{3!}C_1$$

$$C_4 = -\frac{-(n-2)(n+3)}{4.3}C_2 \qquad\qquad C_5 = \frac{(n-3)(n+4)}{5.4}C_3$$

$$= \frac{(n-2)(n)(n+1)(n+3)}{4!}C_0 \qquad\qquad = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} \quad C_1$$

C_6 = ... so on $\qquad$ C_7 = ... so on

Putting the values in equation (2.3.3) we get,

$$u(x) = c_0 u_1(x) + c_1 u_2(x) \text{ ....} \rightarrow \text{ (2.3.6)}$$

Where, $u_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 + \text{......................} \rightarrow$ (2.3.7)

and $u_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 + \text{..............} \rightarrow$ (2.3.8)

Equation (2.3.7) and equation (2.3.8) contain even power of x and odd power of x, respectively.

Legendre Polynomials $P_n(x)$:

The polynomials $u_1(x)$ and $u_2(x)$, multiplied by some constants, are called Legendre polynomials and are denoted by $P_n(x)$. The coefficient of c_n of the highest power x^n is

$$c_n = \frac{(2n)!}{2^n (n!)^2} = \frac{1.3.5\text{................}(2n-1)}{n!} \qquad \text{.......} \rightarrow \text{ (2.3.9)}$$

From equation (2.3.5) we get,

$$c_s = -\frac{(s+2)(s+1)}{(n-s)(n+s+1)}c_{s+2} \qquad \text{.....} \rightarrow \text{ (2.3.10)}$$

Put s + 2 = n, in equation (2.3.10), then

$$C_{n-2} = \frac{n(n-1)}{2(2n-1)} C_n$$

$$= \frac{n(n-1)}{2(2n-1)} \frac{(2n)!}{2^n (n!)^2} \text{ [from equation (2.3.9)]}$$

$$= \frac{n(n-1)2n(2n-1)(2n-2)!}{2(2n-1)2^n n(n-1)!n(n-1)(n-2)!}$$

$$= \frac{(2n-2)!}{2^n (n-1)!(n-2)!}$$

Similarly, $C_{n-4} = -\frac{(n-2)(n-3)}{4(2n-3)} C_{n-2}$

$$= \frac{(2n-4)!}{2^n 2!(n-2)!(n-4)!} \text{ and so on}$$

In general, $n - 2m > 0$,

$$C_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}$$

Putting the value of C_{n-2m} in equation (2.3.3) we obtain the power series solution

$$u(x) = \sum_{m=0}^{\infty} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m} \quad ..... \rightarrow (2.3.11)$$

The resulting solution of equation (2.3.11) is called the Lengendre polynomial of degree n and is denoted by

$$P_n(x) = \frac{(2n)!}{2^n (n!)^2} x^n - \frac{(2n-2)! \times x^{n-2}}{2^n (n-1)!(n-2)!} + \text{..........} \rightarrow (2.3.12)$$

The first few of these function

$$\begin{cases} P_0(x) = 1 \\ P_2(x) = \frac{1}{2}\left(3x^2 - 1\right) \\ P_4(x) = \frac{1}{6}\left(35x^4 - 30x^2 + 3\right) \end{cases} \qquad \begin{cases} P_1(x) = x \\ P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right) \\ P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right) \end{cases}$$

.... (2.3.13)

A few of these function are shown in Fig. 2.1

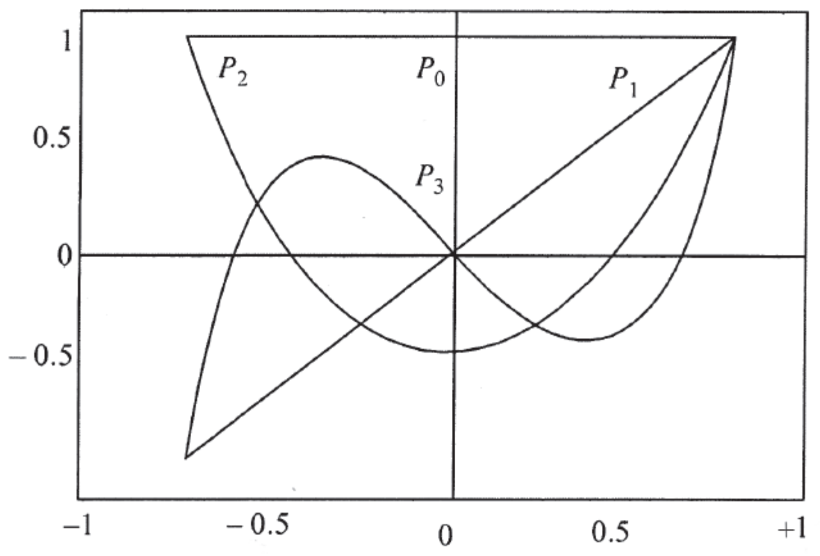


Fig. 2.1 : Legendre function of the first kind $P_n(x)$

2.4 Rodrigue's formula for Legendre Polynomial :

Rodrigue's formula is given for $P_n(x)$ by

$$P_n(x) = \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n}\left(x^2 - 1\right)^n \quad \ldots . \rightarrow (2.4.1)$$

Proof : Let, $t = (x^2 - 1)^n$

$$\Rightarrow \frac{dt}{dx} = n\left(x^2 - 1\right)^{n-1} 2x \quad \ldots . \rightarrow (2.4.2)$$

or, $$\left(x^2 - 1\right)\frac{dt}{dx} = 2ntx \quad \ldots . \rightarrow (2.4.3)$$

Differentiating (2.4.2) by Leibnitz theorem, $(n + 1)$ times $\left(D \equiv \frac{d}{dx}\right)$,

$$\left(x^2 - 1\right)D^{n+2}t + {}^{n+1}C_1(2x)D^{n+1}t + 2^{n+1}C_2D^nt$$

$$= 2n\left\{x\, D^{n+1}t + {}^{n+1}C_1D^nt\right\}$$

or, $$\left(x^2 - 1\right)D^{n+2}t + 2x\left[{}^{n+1}C_1 - n\right]D^{n+1}t + 2\left[{}^{n+1}C_2 - n\ {}^{n+1}C_1\right]D^nt = 0$$

or, $$\left(x^2 - 1\right)D^{n+2}t + 2x\,D^{n+1}t - n(n+1)D^nt = 0$$

or, $$\left(x^2 - 1\right)\frac{d^2y}{dx^2} + 2x\cdot\frac{dy}{dx} - n(n+1)\,y = 0, \quad [\text{Put } D^nt = y]$$

The above equation is Legendre equation.

Hence $y = P_n(x)$ is also its solution.

Now from equation (2.3.1) we can write

$$P_0(x) = 1, P_1(x) = \frac{1}{2}\frac{d}{dx}\left(x^2 - 1\right) = x,$$

$$P_2(x) = \frac{1}{2^2 2!}\frac{d^2}{dx^2}\left(x^2 - 1\right)^2$$

$$= \frac{1}{2^2.2}\cdot\frac{d}{dx}\left[2\left(x^2 - 1\right)2x\right] = \frac{1}{2}\left(3x^2 - 1\right)$$

$$P_3(x) = \frac{1}{2^3 3!}\frac{d^3}{dx^3}\left(x^2 - 1\right)^3 = \frac{1}{2}\left(5x^3 - 3x\right),$$

$$P_4(x) = \frac{1}{2^4 \cdot 4!}\frac{d^4}{dx^4}\left(x^2 - 1\right)^4 = \frac{1}{6}\left(35x^4 - 30x^2 + 3\right) \text{ and so on.......}$$

All the above solutions are matched with the equation (2.3.13)

2.5 Generating function of $P_n(x)$:

$$\left(1 - 2xy + y^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} y^n P_n(x) \text{..........} \quad (2.5.1)$$

Proof : $\left(1 - 2xy + y^2\right)^{-\frac{1}{2}} = \left[1 - y(2x - y)\right]^{-\frac{1}{2}}$

$$= 1+\frac{y}{2}(2x-y)+\frac{1.3}{2.4}y^2(2x-y)^2+\frac{1.3.5}{2.4.6}y^3(2x-y)^3+$$

$$..........+........ \quad +\frac{1.3.5.(2n-1)}{2.4.6..........2n}y^n(2x-y)^n$$

The co-efficients of y^n are

$$\frac{1.3.5..........(2n-1)}{2.4.6.........2n}(2x)^n-\frac{1.3.5..........(2n-3)}{2.4.6...........(2n-2)}(2x)^{n-2}\,{}^{n-1}C_1 \;+\; ....$$

$$= \frac{1.3.5..........(2n-1)}{n!}\left[x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}+.........\right]$$

$= P_n(x)$

$$\left(1-2xy+y^2\right)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}y^nP_n(x)=P_0(x)+yP_1(x)+y^2P_2(x)+ ....$$

$\left(1-2xy+y^2\right)^{-\frac{1}{2}}$ is thus the generating function for Legendre polynomial $P_n(x)$.

2.6 Orthogonality of Legendre Polynomials :

Legendre's polynomials are a set of orthogonal functions in the interval [– 1,1].

$P_n(x)$ is a solution of

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0 \qquad ...\ (2.6.1)$$

$P_m(x)$ is a solution of

$$\left(1-x^2\right)\frac{d^2t}{dx^2}-2x\frac{dt}{dx}+m(m+1)t=0 \qquad \text{... (2.6.2)}$$

Multiplying (2.6.1) by t and (2.6.2) by y and substracting,

We get,

$$\left(1-x^2\right)\left[t\frac{d^2y}{dx^2}-y\frac{d^2t}{dx^2}\right]-2x\left[t\frac{dy}{dx}-y\frac{dt}{dx}\right]+\left[n(n+1)-m(m+1)\right]yt=0$$

By integrating *w. r. t.* x from – 1 to + 1, we get,

$$\left(1-x^2\right)\left(t\frac{dy}{dx}-y\frac{dt}{dx}\right)_{-1}^{1}+(n-m)(n+m+1)\int_{-1}^{+1} y\ tdx=0$$

or, $(n-m)(n+m+1)\int_{-1}^{+1} P_n(x)P_m(x)dx=0$

[Since, y and t are the solutions of Legendre's differential equation]

Hence, $\boxed{\int_{-1}^{+1} P_n(x)P_m(x)dx=0}$ ... (2.6.3)

The above equation is the required condition of orthogenality

2.7 Recurrence formulae for $P_n(x)$:

From equation (2.5.1) we get

$$\left(1-2xy+y^2\right)^{-\frac{1}{2}}=\sum_{x=0}^{\infty} y^n P_n(x)$$

Let, $\left(1-2xy+y^2\right)^{-\frac{1}{2}}=t$ ... (2.7.1)

or, $t^2\left(1-2xy+y^2\right)=1$

Differentiating *w. r. t y* we get,

$$2t\frac{dt}{dy}\left(1-2xy+y^2\right)+t^2\left(-2x+2y\right)=0$$

or, $\frac{dt}{dy}\left(1-2xy+y^2\right)+t\left(y-x\right)=0$

or, $\left(1-2xy+y^2\right)\sum_{n=0}^{\infty} ny^{n-1}P_n\left(x\right)+\left(y-x\right)\sum_{n=0}^{\infty} y^nP_n\left(x\right)=0$ $\left[\text{Put } t=\sum_{n=0}^{\infty} y^nP_n\left(x\right)\right]$

The co-efficients of y^{n-1} from both sides given

$$nP_n\left(x\right)-2x\left(n-1\right)P_{n-1}\left(x\right)+\left(n-2\right)P_{n-2}\left(x\right)+P_{n-2}\left(x\right)-xP_{n-1}\left(x\right)=0$$

or, $\boxed{nP_n\left(x\right)=\left(2n-1\right)xP_{n-1}\left(x\right)-\left(n-1\right)P_{n-2}\left(x\right)}$ ......(2.7.2)

The above equation is the recurrence formulae for $P_n(x)$.

2.8 Bessel's equation :

Bessel's equation of order n is expresed as $x^2\frac{d^2u}{dx^2}+x\frac{du}{dx}+\left(x^2-n^2\right)u=0$ ... (2.8.1)

$x = 0$ is a regular singular point. Therefore power series solution can be written as equation (2.2.4) i.e.

$$u = \sum_{r=0}^{\infty} C_r\, x^{m+r} \qquad \text{... (2.8.2)}$$

Put the value of (2.8.2) in equation (2.8.1) we get,

$$x^2 \sum_{r=0}^{\infty} C_r (m+r)(m+r-1) x^{m+r-2} + x \sum_{r=0}^{\infty} C_r (m+r) x^{m+r-1} + \left(x^2 - n^2\right) \sum_{r=0}^{\infty} C_r x^{m+r} = 0$$

or, $\sum_{r=0}^{\infty} C_r \left[(m+r)(m+r-1) + (m+r) - n^2\right] x^{m+r} + \sum_{r=0}^{\infty} C_r x^{m+r+2} = 0$

or, $\sum_{r=0}^{\infty} C_r \left[(m+r)^2 - n^2\right] x^{m+r} + \sum_{r=0}^{\infty} C_r\, x^{m+r+2} = 0$

Putting, r = 0,1 and Equation the co-efficient of x^m and x^{m+1}

we get,

$$C_0\left[m^2 - n^2\right] = 0 \text{ and } C_1\left[(m+1)^2 - n^2\right] = 0 \Rightarrow \left[(m+1)^2 - n^2\right] \neq 0,$$

So $C_1 = 0$

$\Rightarrow m = \pm n,\ C_0 \neq 0$ (indicial equation)

Equating the co-efficient of the general term x^{m+r+2} to zero, we get,

$$C_{r+2}\left[(m+r+2)^2 - n^2\right] + C_r = 0$$

$$\Rightarrow \boxed{C_{r+2} = \frac{-1}{(m+r+2)^2 - n^2} C_r} \qquad \text{... (2.8.3)}$$

The above equation is recurrence relation.

From (2.8.3) we get

$C_1 = C_3 = C_5$ 0

and $C_2 = \dfrac{-1}{(m+2)^2 - n^2} C_0$

$$C_4 = \frac{1}{\left[(m+2)^2 - n^2\right]\left[(m+4)^2 - n^2\right]} C_0 \quad \text{and so on}$$

If we say $u = Au_1(x) + Bu_2(x)$, then for $m = n$, we can write,

$$u_1(x) = C_0 x^n \left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} \cdots\cdots\right]$$

$$= C_0 x^n \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{2^r r! \ldots\ldots 2^r (n+1)(n+2) \ldots\ldots (n+r)} \qquad \ldots (2.8.4)$$

The solution (2.8.4) is called the Bessel function of the first kind of order n and is denoted by

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} \qquad \ldots (2.8.5)$$

Replacing n by $-n$ in equation (2.8.5) we get,

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)r}{r!\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{-n+2r} \qquad \ldots (2.8.6)$$

The solution (2.8.6) is called the Bessel function of the first kind of order $(-n)$.

Hence the complete solution of equation (2.8.1) is

$$\boxed{U = AJ_n(x) + B.J_{-n}(x)}$$

2.9 Recurrence formula for Bessel functions :

Formula : 1. $xJ'_n = nJ_n - xJ_{n+1}$

Proof : We have,

$$J_n{}' = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

Differentiating *w. r. t. x*, we get,

$$J_n' = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{r!\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \times \frac{1}{2}$$

$$\therefore x J_n{}' = n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} + x \sum_{r=0}^{\infty} \frac{(-1)^r\, 2r}{2.r!\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= nJ_n + x \sum_{r=1}^{\infty} \frac{(-1)^r}{(r-1)!\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= nJ_n + x \sum_{t=1}^{\infty} \frac{(-1)^{t+1}}{t!\Gamma(n+t+2)} \left(\frac{x}{2}\right)^{n+2t+1} \quad \text{[Put, } r - 1 = t\text{]}$$

$$= nJ_n + x \sum_{t=0}^{\infty} \frac{-(-1)^t}{t!\Gamma[(n+1)+t+1]} \left(\frac{x}{2}\right)^{(n+1)+2t}$$

Here $\boxed{xJ_n{}' = nJ_n - xJ_{n+1}}$ ... (2.9.1)

Formulae : 2. $xJ_n{}' = -nJ_n + xJ_{n-1}$

Proof : We have, $J_n{}' = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{r!\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2}$

Hence, $$x J_n{}' = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{r!\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (2n+2r-n)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (2n+2r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} - n\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{2(-1)^r (n+r)}{r!\,(n+r)\Gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r} - nJ_n$$

$$= \sum_{r=0}^{\infty} \frac{2(-1)^r}{r!\,\Gamma(n+r)} \left(\frac{x}{2}\right) \cdot \left(\frac{x}{2}\right)^{n+2r-1} - nJ_n$$

$$= x\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma\left[(n-1)+r+1\right]} \left(\frac{x}{2}\right)^{\left[(n-1)-2r\right]} - nJ_n$$

$\boxed{x\,J_n' = xJ_{n-1} - nJ_n}$ ... (2.9.2)

Formula 3.

From equation (2.9.1) and (2.9.2) we get,

$$2xJ_n' = xJ_{n-1} - nJ_n + nJ_n - xJ_{n+1}$$

Hence $\boxed{2J_n' = J_{n-1} - J_{n+1}}$ ... (2.9.3)

Formula 4. Equating (2.9.1) and (2.9.2) we obtain

$$xJ_{n-1} - nJ_n = nJ_n - xJ_{n+1}$$

$$\boxed{2nJ_n = x\left(J_{n-1} + J_{n+1}\right)} \qquad \text{... (2.9.4)}$$

Formula 5. Multiplying both sides by x^{-n-1} of equation (2.9.1)

We get, $x^{-n}J_n' = nx^{-n-1}J_n - x^{-n}J_{n+1}$

or, $x^{-n}J_n' - nx^{-n-1}J_n = -x^{-n}J_{n+1}$

Hence, $\boxed{\frac{d}{dx}\left(x^{-n}J_n\right) = -x^{-n}J_{n+1}}$ ... (2.9.5)

Formula 6. Multiplying bothsides by x^{n-1} of equation (2.9.2)

We get, $x^nJ_n' = -nx^{n-1}J_n + x^nJ_{n-1}$

or, $xJ_n' + nx^{n-1}J_n = x^nJ_{n-1}$

Hence, $\boxed{\frac{d}{dx}\left(x^nJ_n\right) = x^nJ_{n-1}}$ .....(2.9.6)

2.10 Generating function for $J_n(x)$:

The generating function of Bessel's functions is given by

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x) \qquad \text{... (2.10.1)}$$

Proof : We have, $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = e^{\frac{xt}{2}} \cdot e^{-\frac{x}{2t}}$

$$= \left[1 + \frac{xt}{2} + \frac{1}{2!}\left(\frac{xt}{2}\right)^2 + \ldots\ldots\right] \times \left[1 - \frac{x}{2t} + \frac{1}{2!}\left(\frac{x}{2t}\right)^2 + \ldots\ldots\ldots\right] \qquad \text{... (2.10.2)}$$

The co-efficients of t^n and t^{-n} are given by

$$\frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{-x+2} + \ldots\ldots\ldots = J_n(x) \text{ [from (2.8.5)}$$

and $\dfrac{1}{(-n)!}\left(\dfrac{x}{2}\right)^{-n} - \dfrac{1}{(-n+1)!}\left(\dfrac{x}{2}\right)^{-n+2} + \ldots\ldots\ldots = J_{-n}(x)$ [From (2.8.6)]

Hence, $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = J_0(x) + tJ_1(x) + \ldots\ldots + t^{-1}J_{-1}(x) + \ldots\ldots\ldots$

or, $\boxed{e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)}$ **(Proved)**

Thus Bessel's functions can be derived from the co-efficients of different power of t of equation (2.10.1)

2.11 Zeros of Bessel's Function :

From equation (2.8.5) we obtain

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

All zero of $J_n(x)$,except $x = 0$, are simple. Specifically it states that for any integers $n \geq 0$ and $m \geq 1$, the functions $J_n(x)$ and $J_{n+m}(x)$ have no common zeroes other than the one at $x = 0$.

For $n = 0$, and $m = 1$

$$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r+1)} \left(\frac{x}{2}\right)^{2r} \qquad \text{... (2.11.1)}$$

and $$J_1(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r+2)} \left(\frac{x}{2}\right)^{2r+1} \qquad \text{... (2.11.2)}$$

from equation (2.11.1) we get,

$$J_0(x) = 1 - \frac{1}{1!}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 + \text{.........}$$

From equation (2.11.2) we get,

$$J_1(x) = \frac{x}{2} - \frac{1}{1!\,2!}\left(\frac{x}{2}\right)^3 + \frac{1}{2!\,3!}\left(\frac{x}{2}\right)^5 - \text{........}$$

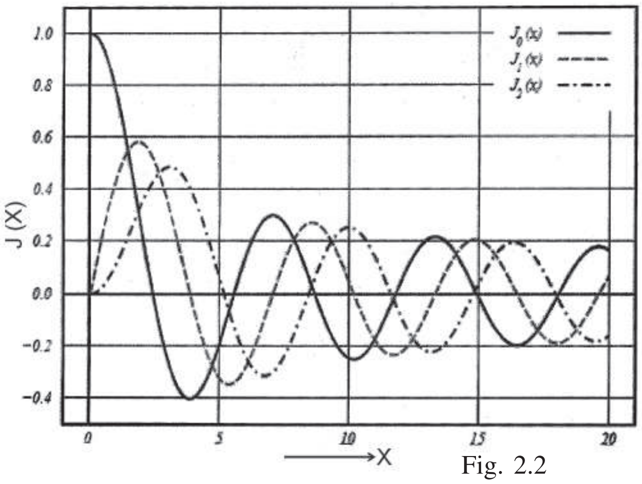


Fig. 2.2

The graphs of the two functions are illustrated in Fig. 2.2. It shows that $J_0(x)$ and $J_1(x)$ have no common zeros.

2.12 Orthogonalities of Bessel functions :

The Bessels equation can be written as

$$x^2 \frac{d^2\Psi}{dx^2} + x\frac{d\Psi}{dx} + \left(\alpha^2 x^2 - n^2\right)\Psi = 0 \qquad \text{... (2.12.1)}$$

and $$x^2 \frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx} + \left(\beta^2 x^2 - n^2\right)\phi = 0 \qquad \text{... (2.12.2)}$$

The solution of equation (2.12.1) is $\psi = J_n(\alpha x)$

The solution of equation (2.12.2) is $\phi = J_n(\beta x)$

Multiply (2.12.1) by $\frac{\phi}{x}$ and (2.12.2) by $\frac{\Psi}{x}$ and subtracting we get,

$$x\left[\phi\frac{d^2\Psi}{dx^2}-\Psi\frac{d^2\phi}{dx^2}\right]+\left[\phi\frac{d\psi}{dx}-\Psi\frac{d^2\varphi}{dx^2}\right] + \left(\alpha^2-\beta^2\right)x\phi\Psi=0$$

or, $\frac{d}{dx}\left[x\left(\phi\frac{d\Psi}{dx}\right)-\Psi\frac{d\phi}{dx}\right]+\left(\alpha^2-\beta^2\right)x\phi\Psi=0$

Integrating both sides w. r. t. x from 0 to 1.

$$x\left(\phi\frac{d\Psi}{dx}-\Psi\frac{d\phi}{dx}\right)\Bigg|_0^1+\left(\alpha^2-\beta^2\right)\int_0^1 x\phi\psi dx=0$$

or, $\left(\alpha^2-\beta^2\right)\int_0^1 x\phi\Psi dx=\left(\phi\frac{d\psi}{dx}-\psi\frac{d\phi}{dx}\right)\Bigg|_0^1$

or, $\int\limits_0^1 x\phi\Psi\, dx=\dfrac{\left(\phi\frac{d\Psi}{dx}-\Psi\frac{d\phi}{dx}\right)\Big|x=1}{\alpha^2-\beta^2}$

or, $\int_0^1 x\phi\Psi\, dx=\dfrac{\left(\phi\frac{d\psi}{dx}-\psi\frac{d\phi}{dx}\right)\Big|_{x=1}}{\left(\alpha^2-\beta^2\right)}$

$$=\frac{\alpha J_n'(\alpha)J_n(\beta)-\beta J_n'(\beta)J_n(\alpha)}{\alpha^2-\beta^2}$$

$\therefore\ \psi=\mathrm{J}_n(\alpha\, x)\ \ \therefore\ \frac{d\psi}{dx}=\alpha J_n'(\alpha x)$ and $\phi=J_n(\beta x)$.

$\therefore\ \frac{d\phi}{dx}=\beta J_n'(\beta x)$

As α, β are distinct roots of $J_n(x) = 0$, then $J_n(\alpha) = J_n(\beta) = 0$, Hence

$\boxed{\int\limits_0^1 x\, J_n(\alpha x)J_n(\beta x)dx=0}$, This is known as orthogonlity relation of Bessel functions.

2.13 Hermite's equation

The Hermiti's equation is give by

$$\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + 2nu = 0 \qquad \text{... (2.13.1)}$$

Here n is positive constant.

Comparing with equation (2.2.1) we obtain $P(x)|_{x=0} \neq 0$ i.e., $x = 0$ is ordinary point and solution can be expressed in the form of equation (2.2.3) i.e.

$$u = \sum_{m=0}^{\infty} C_m\, x^m \qquad \text{... (2.13.2)}$$

Putting the values of equation (2.13.2) in equation (2.13.1)

We get,

$$\sum_{m=0}^{\infty} C_m\, m(m-1)x^{m-2} - \sum_{m=0}^{\infty} 2C_m\, mx^m + \sum_{m=0}^{\infty} 2nC_m x^m = 0 \qquad \text{... (2.13.3)}$$

To obtain the same power of x, we can express the equation (2.13.3) in the form

$$\sum_{s=0}^{\infty} C_{s+2}(s+2)(s+2)x^5 - \sum_{s=0}^{\infty} 2C_s s x^s + \sum_{s=2}^{\infty} 2nC_s x^s = 0 \qquad \text{... (2.13.4)}$$

From equation (2.13.4) we get,

$$C_{S+2} = \frac{2(s-n)}{(s+2)(s+1)}C_s \qquad \text{... (2.13.5)}$$

The equation (2.13.5) is called the recursion or recurrence relation between the co-efficients. Equation (2.13.5) gives us

$$C_2 = \frac{(-2)n}{2.1} C_0\,;\ C_4 = \frac{2(2-n)}{4.3}\,,\ C_2 = -\frac{2(2-n)2n}{4.3.2.1} C_0$$

or, $$C_4 = (-2)^2 \frac{(n-2)n}{4!} C_0,\ C_6 = (-2)^3 \frac{n(n-2)(n-4)}{6!} C_0$$

Similarly,

$$C_3 = \frac{2-2n}{3.2} C_1 = -\frac{2(n-1)}{3!} C_1 = (-2)^1 \frac{(n-1)}{3!} C_1$$

$$C_5 = \frac{6-2n}{5.4} C_3 = -\frac{2(n-1)(6-2n)}{5.4.3!} C_1$$

$$= (-2)^2 \frac{(n-1)(n-3)}{5!} C_1$$

$$C_7 = \frac{2.5-2n}{7.6} = (-2)^3 \frac{(n-5)(n-3)(n-1)}{7.6.5!} C_1$$

$$= (-2)^3 \frac{(n-5)(n-3)(n-1)}{7!} C_1$$

$$= (-2)^3 \frac{(n-1)(n-3)(n-5)}{7!} C_1$$

For even integer, the general from of the co-efficients are ginen by

$$C_{2m} = (-2)^m \frac{n(n-2)\,.........\,(n-2m+2)}{(2m)!} C_0 \qquad ...\ (2.13.6)$$

and for odd integer

$$C_{2m+1} = (-2)^m \frac{(n-1)(n-3)\,........\,(n-2m+1)}{(2m+1)!} C_1 \ ...\ (2.13.7)$$

The solutions (2.13.2) becomes

$$u = \sum_{m=0}^{\infty} C_m x^m$$

$$= c_0 + c_1 x^1 + c_2 x^2 + c_3 x^3 + \ldots\ldots\ldots\ldots$$

$$= \left[c_0 + c_2 x^2 + \ldots\ldots\ldots\right] + \left[c_1 x + c_3 x^3 + \ldots\ldots\ldots\right]$$

$$= \left[1 + (-2)\frac{n}{2!}x^2 + (-2)^2 \frac{n(n-2)}{4!}x^4 + \ldots\right]c_0$$

$$+ \left[x + (-2)\frac{n-1}{3!}x^3 + (-2)^2 \frac{(n-1)(n-3)}{5!}x^5 + \ldots\right]c_1 \qquad \ldots(2.13.8)$$

$$= c_0 u_1(x) + c_1 u_2(x)$$

$$u_1(x) = 1 + (-2)\frac{n}{2!}x^2 + (-2)^2 \frac{n(n-2)}{4!}x^4 + \ldots\ldots \qquad \ldots\ (2.13.9)$$

and $u_2(x) = x + (-2)\frac{n-1}{3!}x^3 + (-2)^2 \frac{(n-1)(n-3)}{5!}x^5 + \ldots\ldots\ldots\ldots\ldots$ (2.13.10)

The equation (2.13.8) is the power series solution of the Hermite's differential equation.

2.14 Hermite polynomial $H_n(x)$:

When n is even, then equation (2.13.8) becomes $u = c_0 u_1(x)$

$$= \left[1 + (-2)\frac{n}{2!}x^2 + \ldots\ldots\ldots + (-2)^{n/2}\frac{n(n-2)\ldots\ldots\ldots 2}{n!}x^n + \ldots\ldots\right]C_0 \ \ldots\ (2.14.1)$$

Let us put $c_0 = (-1)^{n/2}\dfrac{n!}{\left(\frac{n}{2}!\right)}$, then

co-efficient of x^n of equation (2.14.1) given

$$(-1)^{n/2}\frac{n!}{\left(\frac{n}{2}\right)!}\cdot(-2)^{n/2}\frac{n(n-2)......2}{n!}$$

$$= (2)^{n/2}\frac{(2)^{\frac{n}{2}}n(n-2).........2}{\left(\frac{n}{2}\right)!\,2^{n/2}}$$

$$= 2^n\frac{\left[\frac{n}{2}\cdot\left(\frac{n}{2}-1\right)............2\right]}{\left(\frac{n}{2}\right)!} = 2^n$$

Similarly, the co-efficients of x^{n-2} is given by

$$-\frac{n(n-1)}{1!}2^{n-2}$$

$$\therefore \quad u_n(x) = (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} + ... \qquad (2.14.2)$$

The equation (2.14.2) is known as Hermite polynomial of degree n and is denoted by $H_n(x)$. In general,

$$H_n(x) = \sum_{s=0}^{n/2}(-1)^s\frac{n!}{s!(n-2s)!}(2x)^{n-2s}, \; n \text{ is even}$$

and $$H_n(x) = \sum_{s=0}^{n/2-1}(-1)^s\frac{n!}{s!(n-2s)!}(2x)^{n-2s}, \; n \text{ is odd} \quad ... (2.14.3)$$

2.15.1 Generating function of $H_n(x)$:

The generating function of $H_n(x)$ is given by

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) \qquad \text{... (2.15.1)}$$

Proof : Here, $e^{2tx-t^2} = 1+(2tx-t^2)\frac{1}{1!}+\left(2tx-t^2\right)^2 \frac{1}{2!}+\frac{\left(2tx-t^2\right)^3}{3!}$.............

$= 1 + 2t_x - t^2 + \frac{4t^2x^2 - 4t^3x + t^4}{2} + \text{..........}$

$= 1+2tx - t^2 + 2t^2x^2 - 2t^3 x + \frac{t^4}{2} + \text{.......}$

$= 1 + 2xt + (2x^2 - 1)t^2 + \text{...........}$

Now, $\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = H_0(x)+\frac{t}{1!}H_1(x)+ \frac{t^2}{2!}H_2(x)+$.................

$= 1+t(2x)+\frac{t^2}{2!}\left(4x^2-2\right)+\text{.........}$ [from equation 2.14.3]

$= 1 + 2xt + (2x^2 - 1)t^2 + \text{...............}$

Hence, $e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$ **(Proved)**.

2.16 Rodrigue's formula of $H_n(x)$:

The Rodrigue's formula of $H_n(x)$ is given by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right) \qquad \text{... (2.16.1)}$$

Proof : From equation (2.14.3) we obtain,

$H_0(x) = 1$

$H_1(x) = 2x$

$H_2(x) = (2x)^2 - 2!\ (2x)^{2-2}$ [equation (2.14.3) is valid for $s = 0$ and 1]

$= 4x^2 - 2$

$H_3(x) = (2x)^3 - 3!\ (2x)^1$

$= 8x^3 - 12x$ [equation. (2.14.3) is valid for s = 0, 1]

The left hand side of equation (2.16.1) is

for $n = 0$, $(-1)^0\ e^{x^2} \frac{d^\circ}{dx^\circ}\left(e^{-x^2}\right) = 1 = H_0(x)$

for $n = 1$, $(-1)^1 e^{x^2} \frac{d}{dx}\left(e^{-x^2}\right) = 2x = \mathrm{H}_1(x)$

for $n = 2$, $(-1)^2\ e^{x^2} \frac{d^2}{dx^2}\left(e^{-x^2}\right) = 4x^2 - 2 = H_2(x)$

for $n = 3$, $(-1)^3 e^{x^2} \frac{d^3}{dx^3}\left(e^{-x^2}\right)$

$$= -e^{x^2} \frac{d^2}{dx^2}\left(-2x\, e^{-x^2}\right)$$

$$= -e^{x^2} \frac{d}{dx}\left(4x^2 e^{-x^2} - 2e^{-x^2}\right)$$

$$= -e^{x^2}\left[-8x^3 e^{-x^2} + 8xe^{-x^2} + 4xe^{-x^2}\right]$$

$= 8x^3 - 12x = H_3(x)$

Hence, the equation (2.16.1) is proved.

2.17 Recurrence formula of $H_n(x)$:

(1) $2nH_{n-1}(x) = H'n(x)$

We have the equation (2.14.1)

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$$

Differentiating both sides *w. r. t x* we obtain

$$2te^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n'(x)$$

$$\Rightarrow 2t\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(n) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n'(x)$$

$$\Rightarrow 2t\sum_{n=0}^{\infty} \frac{t^{n+1}}{n!} H_n(x) = \sum H_n'(x)\frac{t^n}{n!}$$

The co-efficients of t^n gives

$$\frac{2H_{n-1}(x)}{(n-1)!} = \frac{H_n'(x)}{n!}$$

or, $$2\frac{n!}{(n-1)!} H_{n-1}(x) = H_n'(x)$$

$$\boxed{\therefore\ 2nH_{n-1}(x) = H_n'(x)} \qquad \dots (2.17.1)$$

(2) $H_{n+1}(x)-2xH_n(x)+2nH_{n-1}(x)=0$

Proof : From equation (2.16.1) we obtain,

$$H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

$$\therefore \quad H_n'(x)=(-1)^n 2x\, e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

$$+\ (-1)^n e^{x^2}\frac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right)$$

or, $$H_n'(x)=2x(-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)+(-1)(-1)^{x+1}e^{x^2}\frac{d^{n+1}}{dx^{n+1}}e^{x^2}$$

or, $$2n\, H_{n-1}(x)=2x\, H_n(x)-H_{n+1}(x)$$

$$\therefore H_{n+1}(x)-2x\, H_n(x)+2n\, H_{n-1}(x)=0 \quad \text{... (2.17.2)}$$

3. $H_n''(x)-2x\, H_n'(x)+2nH_n(x)=0$

Proof : We have equation (2.17.1), $H_n'(x)=2n\, H_{n-1}(x)$

$$\therefore \quad H_n''(x)=2n\, H_{n-1}'(x)=\frac{d}{dx}\left[2x\, H_n(x)-H_{n+1}(x)\right] \quad \text{[From 2.17.2]}$$

or, $$H_n''(x)=2\, H_n(x)+2xH_n'(x)-H_{n+1}'(x)$$

$$= 2H_n(x)+2x\, H_n'(x)-2(n+1)H_n(x) \quad \text{[From 2.17.1]}$$

$$= 2xH_n'(x) - 2xH_n(x)$$

$\therefore \quad H_n'(x) - 2x\, H_n'(x) + 2n\, H_n(x) = 0$ [From 2.17.3] **(Proved)**

2.18. Orthogonality of Hemite Polynomials:

If $H_n(x)$ is the solution of equation (2.13.1), then we can write,

$$\frac{d^2H_n(x)}{dx^2} - 2x\frac{dH_n(x)}{dx} + 2nH_n(x) = 0 \quad \text{... (2.18.1)}$$

Multiplying equation (2.18.1) by e^{-x^2} we get,

$$\frac{d}{dx}\left[e^{-x^2}\frac{dH_n(x)}{dx}\right] = -2n\, e^{-x^2} H_n\,(x) \quad \text{... (2.18.2)}$$

and $$\frac{d}{dx}\left[e^{-x^2}\frac{d\,Hm^{(x)}}{dx}\right] = -2m\, e^{-x^2} H_m\,(x) \quad \text{...(2.18.3)}$$

Multiplying equation (2.18.2) by $H_m(x)$ and equation (2.18.3) by $H_n(x)$ and subtracting both equation we get,

$$H_m(x)\frac{d}{dx}\left(e^{-x^2}\frac{dHn(x)}{dx}\right) - H_n(x)\frac{d}{dz}\left(e^{-x^2}\frac{dH_m(x)}{dx}\right)$$

$$= 2(m-n)e^{-x^2}H_m(x)H_n(x) \quad \text{.... (2.18.4)}$$

Integrating both sides $w.\ r.\ t.\ x$ from $-\infty$ to $+\infty$ of eqnation (2.18.4) we obtain,

$$2(\text{m} - \text{n}) \int_{-\infty}^{\infty} e^{-x^2} H_m(x)H_n(x)\,dx$$

$$= \int_{-\infty}^{\infty}\frac{d}{dx}\left\{H_m(x)\left[e^{-x^2}\frac{dH_n(x)}{dx}\right]\right\}dx - \int_{-\infty}^{\infty}\frac{d}{dx}\left\{H_m(x)\left[e^{-x^2}\frac{dH_m(x)}{dx}\right]\right\}dx$$

$= 0 - 0 = 0$

Since, $m \neq n$, then

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x)\,H_n(x) = 0 \quad \text{... (2.18.5)}$$

Equation (2.18.5) slows Hermite polynomials are otthogonal *w.r.t.* e^{-x^2}

2.19 Summary

1. Frobenius method to find out infinite series solutions for a second order differential equation has been discussed.
2. Frobenius—Fuch's theorem to examine whether the power series solution exists or not has been discussed.
3. Steps for solving series solution has been discussed.
4. Legendre differential equation, Bessel's equation, Hermite equation and their generating functions, recurrence formula and orthogonalities have been discussed.

2.20 Review Questions and Answer :

1. What is Frobenius method ?

Ans. The Frobenius means is a method by which one can expand a power series solution to much a diffrential equations of the form $z^2 \dfrac{d^2u}{dz^2} + ZP(z)\dfrac{du}{dz} + q(z)u = 0$, where $p(z)$ and $q(z)$ are analytic at $z = 0$ or being analytic eleswhere.

For more details see the text book.

2. What is singular point in a differential equation?

Ans. Let us consider a second-order differential equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0$$

At any point $x = a$, if $P(x)$ and $Q(x)$ is finite, then $x = a$ is called ordinary point and if $P(x)$ or $Q(x)$ diverges as $x \to a$, then $x = a$ is called singular point.

Suppose $P(x)$ and $Q(x)$ diverges as $x \to a$, but $(x - a)P(x)$ and $(x - a)^2 Q(x)$ remain finite as $x \to a$, then $x = a$ is called regular or nonessential singular point.

If $(x - a) P(x)$ and $(x - a)^2 Q(x)$ remain infinite as $x \to a$, then $x = a$ is called essential singular point or irregular singular point.

See also the article No (2.2) also.

3. What is the indicial equation ?

Ans. An indicial equation is also called a characteristic equation, when we solve a second-order ordinary differtial equation by Frobenius method, a recurrence relation we obtain, which is known as indicial equation

Equation (2.3.5), Equation (2.8.3) are the indicial equations of Legendre and Bessel's equation respectively.

4. Find the series solution of the differencial equation $(1 + x^2) u'' + xu' - u = 0$.

Ans. The given equation is

$$(1 + x^2)u'' + xu' - u = 0 \qquad \text{... (1)}$$

At $x = 0$, $P(x)|_{x=0} \neq 0$ i.e., $x = 0$ is ordinary point of the equtation and series solution have the form $u = \sum_{m=0}^{\infty} C_m x^m$ ... (2)

Putting the value of (x) in equation (1) we get,

$$\left(1+x^2\right) \sum_{m=0}^{\infty} m(m-1)C_m\, x^{m-2} + x \sum_{m=0}^{\infty} C_m\, mx^{m-1} - x \sum_{m=0}^{\infty} C_m x^m = 0$$

or, $$\sum_{m=0}^{\infty} m(m-1)C_m\ x^{m-2} + \sum_{m=0}^{\infty} m(m-1)C_m\ x^m + \sum_{m=0}^{\infty} m\ C_m\, x^m - \sum_{m=0}^{\infty} C_m\ x^m = 0$$

.... (3)

To obtain the same power of x, we can write eqation (3) in the form,

$$\sum_{s=0}^{\infty}(s+2)(s+1)C_{s+2}\,x^s + \sum_{s=0}^{\infty} s(s-1)C_s x^s + \sum_{s=0}^{\infty} sC_s x^s - \sum_{s=0}^{\infty} C_s x^s = 0$$

The co-efficient of x^s is

$(s + 2)(s+1)\ C_{s+2} + (s^2 - s + s - 1)\ C_s = 0$

$$\therefore\ C_{s+2} = \frac{s^2-1}{(s+2)(s+1)}C_s = -\frac{s-1}{s+2}C_s \qquad \text{... (4)}$$

From equation (4) we get the values

$$C_2 = +\frac{1}{2}C_0$$

$$C_4 = \frac{1}{4}C_2 = -\frac{1}{8}C_0$$

$$C_6 = -\frac{3}{6}C_4 = \frac{1}{2}\cdot\left(+\frac{1}{8}\right)C_0 = +\frac{1}{16}C_0$$

C_8 = ... and so on.

$C_3 = 0,\ C_5 = C_7 =$ 0

Putting the values of co-efficients in equation (2) we get the series solution

$u(x) = C_0 + C_1x + C_2x^2 + C_4x^4 + C_6x^6 + \ldots\ldots\ldots\ldots$

$$= C_0\left[1+\frac{1}{2}x^2-\frac{1}{8}x^4+\frac{1}{16}x^6.........\right]+C_1x$$

5. Express $5x^3 + x$ in terms of Legendre Polynomials.

Ans. We have, $P_3(x)=\frac{1}{2}\left(5x^3-3x\right)$

or, $5x^3 = 2P_3(x) + 5x$

$\therefore$ $f(x) = 5x^3 + x = 2P_3(x) + 4x = 2P_3(x) + 4P_1(x)$ [$\because P_1(x) = x$]

6. Express $f(x) = x^3 + 2x^2 - x - 3$ interms of Legendre Polynomials.

Ans. We have, $P_3(x)=\frac{1}{2}\left(5x^3-3x\right)$

or, $x^3=\frac{2}{5}P_3(x)+\frac{3}{5}x$

$$P_2(x)=\frac{1}{2}\left(3x^2-1\right)$$

or, $x^2=\frac{2}{3}P_2(x)+\frac{1}{3}$

$P_1(x) = x$ and $P_0(x) = 1$

$\therefore$ $f(x) = x^3 + 2x^2 - x - 3$

$$= \frac{2}{5}P_3(x)+\frac{3}{5}x+\frac{4}{3}P_2(x)-\frac{2}{3}+x-3$$

$$= \frac{2}{5}P_3(x)+\frac{4}{3}P_2(x)-\frac{2}{5}P_1(x)-\frac{7}{3}P_0(x)$$

7. Prove that $P_n(1) = 1$

Ans. We have

$$\left[1-t(2x-t)\right]^{-1/2}=\sum_{n=0}^{\alpha}P_n(x)t^2$$

Putting $x = 1$, we get,

$$\left[1-2t+t^2\right]^{-1/2} = \sum_{n=0}^{\alpha} P_n(1)t^n$$

or, $$\left[(1-t)^2\right]^{-1/2} = \sum_{n=0}^{\alpha} P_x(1)t^2$$

or, $$\left[(1-t)^2\right]^{-1} = \sum_{n=0}^{\alpha} P_n(1)t^n$$

or, $1 + t + t^2 + \ldots\ldots\ldots + t_n + \ldots\ldots\ldots = 1 + P(1).\mathrm{t} + P_2(t)t^2 + \ldots\ldots + \ldots.P_n(t)t^n + \ldots\ldots$

Equating the co-efficients of t^n an either side we get,

$P_n(1) = 1$

8. Show that $P_n(-x) = (-1)n\ P_n(\mathrm{x})$

Ans. We have $\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^2 P_n(n)$

Putting $x = -x$, we get

$$\left(1+2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(-x) \quad \ldots (1)$$

Also, putting $t = -t$ we get,

$$\left(1+2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} (-1)^n t^2 P_n(x) \quad \ldots (2)$$

From equation (1) & (2) we get,

$$\sum_{n=0}^{\infty} t^n P_n(-x) = \sum_{n=0}^{\infty} (-1)^n t^n P_n(x)$$

Compairing the co-effcients of t^n we get

$P_n(-x) = (-1)^n \; P_n(x)$

9. If $P_n(x)$ is a Legendre Polynomial of degree n and α is is such that $P_n(\alpha)$ 0, then show that $P_{n-1}(\alpha)$ and $P_{n+1}(\alpha)$ are of opposite signs.

Ans. From the recurrence relation we get,

$(2n+1)x\, P_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$ ... (1)

Putting, $x = \alpha$ is equation (1) we get,

$(2n+1)\,\alpha\, P_n(\alpha) = (n+1)P_{n+1}(\alpha) + nP_{n-1}(\alpha)$

or, $0 = (n+1)P_{n+1}(\alpha) + nP_{n-1}(\alpha),.\ P_n(\alpha) = 0$

or, $$\frac{P_{n+1}(\alpha)}{P_{n-1}(\alpha)} = \frac{n}{n+1}$$

As n is positive integer so R. M. S is negative.

Hence $P_{n+1}(\alpha)$ and $P_{n-1}(\alpha)$ are of opposite signs.

10. Show that $\frac{d}{dx}J_0(x) = -J_1(x)$

Ans. We have,

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\Gamma(n+r+1)}$$

Putting $n = 0$

$$J_0(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{2r} \frac{1}{r!\Gamma(r+1)}$$

$$\therefore \quad \frac{d}{dx} J_0(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{2r-1} \frac{r}{r!\Gamma(r+1)}$$

$$= \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{2r-1} \frac{1}{r!\Gamma(r)}$$

$$= J_{-1}(x) = -J_1(x)$$

11. Show that $J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$

Ans. We have, $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$

or, $J_{n+1}(x) = \frac{2n}{x} \cdot J_n(x) - J_{n-1}(x)$

Putting, $n = 1$, $J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$

$n = 2$, $J_3(x) = \frac{4}{x} J_1(x) - J_1(x)$

$n = 3$. $J_4(x) = \frac{6}{x} J_3(x) - J_2(x)$

$$\therefore \quad J_4(x) = \frac{6}{x} J_3(x) - J_2(x)$$

$$= \frac{6}{x}\left[\frac{4}{x} J_2(x) - J_1(x)\right] - \frac{2}{x} J_1(x) + J_0(x)$$

$$= \frac{24}{x^2}J_2(x) - \frac{6}{x}J_1(x) - \frac{2}{x}J_1(x) + J_0(x)$$

$$= \frac{24}{x^2}\left[\frac{2}{x}J_1(x) - J_0(x)\right] - \frac{8}{x}J_1(x) + J_0(x)$$

$$= \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) + \left(1 - \frac{24}{x^2}\right)J_0(x)$$

12. Prove that $\int J_0(x)J_1(n)dx = -\frac{1}{2}[J_0(x)]^2$

Ans. LHS $= \int J_0(x)J_1(x)dx$

$$= \int J_0(x)\left[-\frac{d}{dx}J_0(x)\right]dx \qquad \left[\because J_1(x) = -\frac{d}{dx}J_0(x)\right]$$

$$= -\frac{1}{2}\int \frac{d}{dx}[J_0(x)]^2 dx$$

$$= -\frac{1}{2}[J_0(x)]^2$$ **(Proved)**

13. Prove that $\int x_1 J_0(x^2)dx = \frac{1}{2}x^2\left[J_0(x)^2 + J_1(x)^2\right]$

Ans. $\int xJ_0^2(x)\,dx$

$$= J_0^2(x)\frac{x^2}{2} - \int 2J_0(x)J_0'(x)\frac{x^2}{2}dx \qquad \text{[integrating by parts]}$$

$$= \frac{x^2}{2}J_0{}^2(x) + \int x^2 J_0(x)J_1(x)dx \qquad [\because J_0'(x) = -J_1(x)]$$

$$= \frac{x^2}{2}J_0^2(x) + \int xJ_1\frac{d}{dx}\left[xJ_1(x)\right]dx \qquad \left[\because \frac{d}{dx}\left[xJ_1(x)\right] = xJ_0(x)\right]$$

$$= \frac{x^2}{2} J_0^2(x) + \frac{1}{2}\int \frac{d}{dx}\left[x J_1(x)\right]^2 dx$$

$$= \frac{x^2}{2} J_0^2(x) + \frac{1}{2} x^2 J_1^2(x)$$

$$= \frac{x^2}{2}\left[J_0^2(x) + J_1^2(x)\right] \quad \textbf{(Proved)}$$

14. Show that $\int_0^x x^n \mathrm{J}_{n-1}(x)dx = x^n \mathrm{J}_n(x)$

Ans. We have $\frac{d}{dx}\left[x^n J_n(n)\right] = x^n J_{n-1}(x)$

Integrating *w. r. t. x* between the limits *o* and *x*, we get,

$$\int_o^x \frac{d}{dx}\left[x^n \mathrm{J}_n(x)\right]dx = \int_0^x x^n \cdot J_{n-1}(x)dx$$

or, $x^n J_n(n) = \int_0^x x^n J_{n-1}(x)dx$

or, $\int_0^x x^n J_{n-1}(x)dx = x^n J_n(x)$

Unit–3 ❑ Some Special Integrals

Structure

3.0 Objectives

1. To know what are special functions like Beta and Gamma functions and relation between them.
2. To know how to differentiate indefinite and definite integrals.
3. To make an idea about error function.

3.1 Introduction

Algebraic, trigonometric, exponential, and logarithmic functions are the elementary function. Bessel and gamma functions are the special functions of mathematics. These functions used to solve many types of problems in physics and engineering. We shall study these functions and its properties. There is an important relation between beta and gamma functions.

In mathematics, the error function (also called the Gauss error function), often denoted by erf, is a complex function of a complex variable. This error function is a special (non-elementary) sigmoid function that occurs often in probability, statistics, and partial differential equations. In many of these applications, the function argument is a real number. If the function argument is real, then the function value is also real.

3.2 Beta Function :

The beta function was studied by Euler and Legendre and was given its name by Jacques Binet. Symbolically, it represents by Greek lower case β and defined by

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \text{ where m > 0, n > 0}$$

The beta function is known as Euler's integral of the first kind. ......(3.2.1)

Properties of β function :

1. β function is symmetric i.e.,

$\beta\ (m,\ n) = \beta\ (n,\ m)$ ......(3.2.2)

Proof : From equation : (3.1.1) we obtain

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Let us put $1 - x = y$, then $dx = - dy$

Hence, $$\beta(m,n)=\int_1^0 (1-y)^{m-1}\,y^{n-1}(-dy)$$

$$=\int_0^1 y^{n-1}(1-y)^{m-1}\,dy=\beta(n,m)$$

2. β function can be expressed by

$$\beta\ (m,\ n) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\,d\theta \text{ ....(3.2.3).}$$

Proof : Let us put $x = \sin^2\theta$ in equation (3.1.1), then $dx = 2\sin\theta\cos\theta\ d\theta$ and

$$\beta\ (m,\ n) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-2}(\cos\theta)^{2n-2}\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1} \, d\theta$$

3. $\beta(m, n) = \int_0^\infty \frac{t^{n-1}}{(1+t)^{m+n}} dt$ ......(3.2.4)

Proof : From equation (3.2.1) we get,

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

Let us put $1+t = \frac{1}{x}$

$$\Rightarrow dt = \frac{-1}{(x)^2} dx$$

Hence,
$$\beta(m, n) = -\int_\infty^0 \frac{1}{(1+t)^{m-1}} \frac{t^{n-1}}{(1+t)^{n-1}} \frac{1}{(1+t)^2} dt$$

$$= \int_0^\infty \frac{t^{n-1}}{(1+t)^{m+n}} dt$$

3.3 Gamma function :

Gamma function is represented by capital Greek alphabet, Γ and is define by

$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$ ....(3.3.1), which converges for $n > 0$.

Now, $\Gamma(1) = \int_0^\infty e^{-x} dx = -e^{-x}\Big]_0^\infty = 1$ ...... (3.3.2)

Recursion formula of Γ(n) :

From equation (3.3.1) we, can write,

$$\Gamma(n+1)=\int_0^{\infty} e^{-x}x^n dx, \text{ [Putting } (n+1) \text{ in place of } n]$$

$$= \left. -x^n e^{-x} \right|_o^{\alpha} + n\int_0^{\infty} e^{-x}x^{n-1}dx$$

$$= 0 + n\,\Gamma(n)$$

or, $\boxed{\Gamma(n+1)=n\Gamma(n)}$ ......(3.3.3)

Equation (3.3.3) can be written as

$$\Gamma(n)=\frac{\Gamma(n+1)}{n}$$

$\Gamma(n) \to \infty$, when $n \to 0$

When n $\to -1$, then

$$\Gamma(-1)=\frac{\Gamma(0)}{-1}\to\infty, \text{ as } \Gamma(0) \to \infty, \text{ and so on.}$$

Hence, n should be positive integer and Γ function can not be defined for n = 0, – 1, – 2, ...

i.e., n = 0 and negative integer.

From equation (3.3.3) we obtaion

$\Gamma(2) = 1.\ \Gamma(1) = 1\ !$

$\Gamma\ (3) = 2\ .\ \Gamma\ (2) = 2.1 = 2\ !$

$\Gamma(4) = 3.\ \Gamma(3) = 3.2\ ! = 3\ !$

$\Gamma\ (5) = 4.\ \Gamma\ (4) = 4.3\ ! = 4\ !$

....

....

$\Gamma(n) = (n -1)\ \Gamma\ (n - 1) = (n - 1)\ !$

or, $\Gamma(n + 1) = (n + 1 - 1)\ ! = n\ !$

So, $\Gamma\ (n + 1) = n\ !$ ....(3.3.4), when n is positive integer)

The gamma function can be considered as a factorial function to which it reduces when n as a positive integer. The gamma function is known as Euler's integral of the socond kind.

3.3.1 Value of $\Gamma\left(\frac{1}{2}\right)$:

From equation (3.2.1) we write

$$\Gamma\left(\frac{1}{2}\right)=\int_0^{\infty} e^{-x}x^{-\frac{1}{2}}dx$$

Let us put $x = y^2$

So $dx = 2y\ dy$, therefore

$$=2\int_0^{\infty} e^{-y^2} y^{-1} y\,dx$$

$$= \Gamma\left(\frac{1}{2}\right)=2\int_0^{\infty} e^{-y^2}dy$$

Hence, $$\left[\Gamma\left(\frac{1}{2}\right)\right]^2=\left[2\int_0^{\infty} e^{-y^2}\,dy\right]\left[2\int_0^{\infty} e^{-x^2}\,dx\right]$$

$$= 2\int_0^{\infty}\int_0^{\infty} e^{-\left(x^2+y^2\right)}dx\,dy$$

Now, if we change the above integral into polar coordinate (r, θ), then we can write,

$$x^2 + y^2 = r^2$$

and $$dx\ dy = r\ dr\ d\theta$$

So, $$\left[\Gamma\left(\frac{1}{2}\right)\right]^2=4\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2}r\,dr\,d\theta$$

$$= 2\pi \int_0^\infty re^{-r^2} dr = 2\pi \left[\left(-\frac{1}{2} \right) e^{-r^2} \right]_0^\infty$$

$$= \pi$$

or, $\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$ ....(3.3.5)

3.4 Relation between beta and gamma functions :

We have the equation (3.3.1)

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, n > 0$$

Put $x = y^2$ and $dx = 2y\, dy$, then

$$\Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1} dy$$

So as $\Gamma(m) = 2\int_0^\infty e^{-z^2} z^{2m-1} dz$

Hence, $\Gamma(n)\Gamma(m) = 4\int_0^\infty \int_0^\infty e^{-(y^2+z^2)} y^{2n-1} z^{2m-1} dy\, dz$

Put, $y = r \cos \theta$

$z = r \sin \theta$

It give us the transformation of coordinate system from certesian to polar coordinate, so, we can write

$dy\, dz = r\, dr\, d\theta$

Limit should be r from 0 to ∞

θ from 0 to $\frac{\pi}{2}$

Thus, $$\Gamma(n)\Gamma(m) = 4\int_0^\infty\int_0^{\frac{\pi}{2}} e^{-r^2} r^{2n-1} (\cos\theta)^{2n-1} r^{2m-1} (\sin\theta)^{2m-1} r\, dr\, d\theta$$

$$= 4\int_0^\infty\int_0^{\frac{\pi}{2}} e^{-r^2} r^{2(n+m)-1} (\cos\theta)^{2n-1} (\sin\theta)^{2m-1} dr\, d\theta$$

$$= \left[2\int_0^{\frac{\pi}{2}} (\cos\theta)^{2n-1} (\sin\theta)^{2m-1} d\theta\right] \times$$

$$\left[2\int_0^\infty e^{-r^2} r^{2(n+m)-1} dr\right]$$

$$= \beta(n,m)\, 2\int_0^\infty e^{-r^2} r^{2(n+m)-1} dr$$

Let us put $r^2 = t \Rightarrow 2dr = \frac{dt}{r}$

and $r^{2(n+m)} = t^{n+m}$

$$= \beta(n,m)\int_0^\infty e^{-t} t^{(n+m)} t^{-1} dt$$

$$= \beta(n,m)\int_0^\infty e^{-t} t^{(n+m)-1} dt$$

$$= \beta\ (n,\ m)\ \Gamma\ (n + m)$$

$$\boxed{\therefore \beta(n,m) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}} \quad \text{..... (3.4.1)}$$

The above relation is useful to evaluate the definite integrals by gamma function.

Example 1 : Evaluate $I = \int_0^{\frac{\pi}{2}} \sin^p x \cos^q x\, dx$ ......(3.4.2)

We have from equation ...(3.2.3)

$$\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

Let us write equation (3.4.2) by

$$I = \frac{1}{2}\cdot 2\int_0^{\frac{\pi}{2}} \sin^{2\left(\frac{p}{2}+\frac{1}{2}\right)-1}\theta \cos^{2\left(\frac{q+1}{2}\right)-1}\theta\, d\theta$$

$$= \frac{1}{2}\beta\left(\frac{p+1}{2,}\ \frac{q+1}{2}\right) \text{ [Compairing equation (3.2.3)}$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)} \quad \text{..... (3.4.3)}$$

Let us put $q = 0$ in equation (3.4.2), gives us

$$I = \int_0^{\frac{\pi}{2}} \sin^p x\, dx = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}+1\right)}$$

and also $$\int_0^{\frac{\pi}{2}} \cos^q x\,dx = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2}+1\right)}$$

Example 2 : Relation between β (*n*, *m*) and β {(*n* – 1), (*m* – 1)}

From equation (3.4.1) we obtain,

$$\beta(n,m) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}$$

$$= \frac{(n-1)\Gamma(n-1)(m-1)\Gamma(m-1)}{(n+m-1)\Gamma(n+m-1)} \quad \text{[from equation (3.3.3)]}$$

$$= \frac{(n-1)(m-1)\Gamma(n-1)\Gamma(m-1)}{(n+m-1)(n+m-2)\Gamma(n+m-2)}$$

or, $$\beta(n,m) = \frac{(n-1)(m-1)}{(n+m-1)(n+m-2)}\beta\{(n-1),(m-1)\} \quad \text{...... (3.4.4)}$$

The above equation represents the reduction formula relating β (*n*,*m*) and β (*n*–1, *m* – 1).

Example 3. Evaluate β (*n*, *m* + 1)

$$\beta\ (n,\ m+1) = \frac{\Gamma(n)\Gamma(m+1)}{\Gamma(n+m+1)}$$

$$= \frac{(n)!(m+1)!}{(n+m+1)!}$$

$$= \frac{n!(m+1)!}{n(n+1)...(n+m)(m+1)!}$$

$$= \frac{1.2.3......n}{n(n+1).....(n+m)} \quad \text{........ (3.4.5)}$$

3.5 Differentiation of indefinite integrals :

Let an integrable function $f = f(x,\alpha)$, where α is a variable parameter and we assume that

$$\int f(x,\alpha)dx = F(x,\alpha) \quad ....(3.5.1)$$

$$\Rightarrow \quad \frac{\partial F(x,\alpha)}{\partial x} = f(x,\alpha)$$

or, $$\frac{\partial^2 F(x,\alpha)}{\partial x\,\partial\alpha} = \frac{\partial f(x,\alpha)}{\partial\alpha}$$

Integrating bothsides w.r.t.x we get

$$\int \frac{\partial f(x,\alpha)}{\partial\alpha}dx = \frac{\partial F(x,\alpha)}{\partial\alpha} \quad ...... (3.5.2)$$

Let us take an example : Evaluate $\int xe^{\alpha x}dx$

we know, $\int e^{\alpha x}dx = \frac{1}{\alpha}e^{\alpha x}$

Using equation (3.5.2) we get

$$\int xe^{\alpha x}dx = \frac{\partial}{\partial\alpha}\left(\frac{1}{\alpha}e^{\alpha x}\right)$$

$$= \left(\frac{x}{\alpha} - \frac{1}{\alpha^2}\right)e^{\alpha x} \quad (3.5.3)$$

In the same manner, we get

$$\int x^2 e^{\alpha x}dx = \left(\frac{x^2}{\alpha} - \frac{2x}{\alpha^2} + \frac{2}{\alpha^3}\right)e^{\alpha x} \quad ...(3.5.4)$$

3.6. Differentiation of definite integrals :

Let the integral be : $I(\alpha)=\int_a^b f(x,\alpha)dx$

where, $f(x, \alpha)$ is an integrable function within the limit $a \leq x \leq b$.

From equation (3.5.1) we get,

$$I(\alpha)=\int_a^b f(x,\alpha)dx = F(x,\alpha)\Big|_a^b$$

$$= F(b, \alpha) - F(a, \alpha) \qquad \text{... (3.6.1)}$$

or,

$$\int_a^b \frac{\partial f(x,\alpha)}{\partial\alpha}dx = \frac{\partial F(b,\alpha)}{\partial\alpha} - \frac{\partial F(a,\alpha)}{\partial\alpha} \qquad \text{...(3.6.2)}$$

Differentiating w.r.t. α, equation (3.6.1) we get,

$$\frac{dI(\alpha)}{d\alpha} = \frac{\partial F(b,\alpha)}{\partial b}\frac{db}{\partial\alpha} + \frac{\partial F(b,\alpha)}{\partial\alpha} - \frac{\partial F(a,\alpha)}{\partial a}\frac{da}{d\alpha} - \frac{\partial F(a,\alpha)}{\partial\alpha}$$

$$= f(b,\alpha)\frac{db}{\partial\alpha} - f(a,\alpha)\frac{da}{d\alpha} + \int_a^b \frac{\partial f(x,\alpha)}{\partial\alpha}dx \qquad \text{...(3.6.3)}$$

[using equation (3.5.2)]

If the limits of integration be independent of α, then

$$\frac{d}{d\alpha}\left[\int_a^b f(x,\alpha)dx\right] = \int_a^b \frac{\partial f(x,\alpha)}{\partial\alpha}dx \qquad \text{...(3.6.4)}$$

Equation (3.6.4) is known as Leibnitz rule.

Let us take an example,

$$I(\alpha)=\int_0^\infty \frac{e^{-\alpha x}\sin x}{x}dx, \quad \alpha \geq 0$$

or, $$\frac{dI}{d\alpha}=\int_0^\infty \frac{\partial}{\partial\alpha}\left(\frac{e^{-\alpha x}\sin x}{x}\right)dx \quad \text{[using equation (3.6.4)]}$$

$$= -\int_0^\infty e^{-\alpha x}\sin x\,dx = -\frac{1}{1+\alpha^2} \quad \text{[Integrating by parts]}$$

On integrating both sides we get

$$I(\alpha)=-\tan^{-1}\alpha+c$$

As, $\alpha \to \infty$, I $\to$ 0, $\Rightarrow$ c = $\frac{\pi}{2}$

$$I(\alpha)=\int_0^\infty \frac{e^{-\alpha x}\sin x}{x}dx=\frac{\pi}{2}-\tan^{-1}\alpha=\cot^{-1}\alpha$$

Putting α = 0, we get,

$$\int_0^\infty \frac{\sin x}{x}dx=\frac{\pi}{2} \qquad \text{...(3.6.5)}$$

3.7 Error function :

The error function, (also called probability integral) is defined as

or, $$erf(x)=\frac{1}{\sqrt{\pi}}\int_{-x}^{x} e^{-t^2}dt$$

$$= \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \qquad \text{.... (3.7.1)}$$

The error function is called the Gauss error function is a special function of sigmoid shape that occurs in probability, statistics and partial differential equations. It is shown in Fig. 3.1.

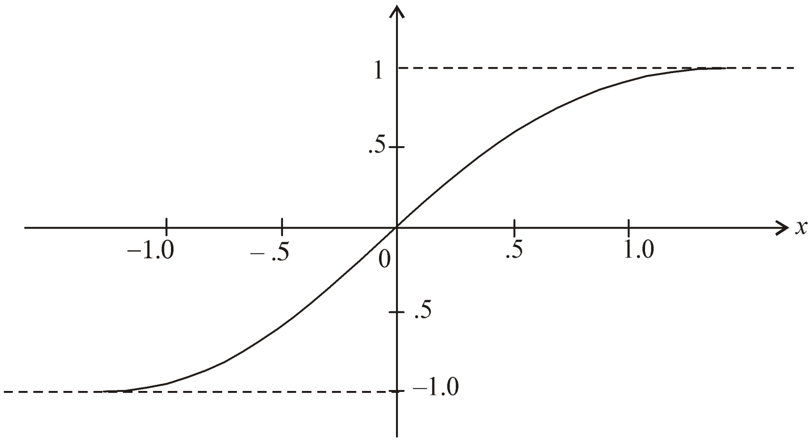


Fig. 3.1

Properties : The properties of error function is give below :

1. $erf\,(x) = -\,erf\,(-x)$,
2. $erf\,(0) = 0$,
3. $erf\,(\infty) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}dt = \frac{2}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2} = 1$

The complementary error function is defined as

$$erf_c\left(x\right) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt \qquad ...(3.7.2)$$

From equation (3.7.1) and equation (3.7.2) we get

$$erf\left(x\right) + erf_c\left(x\right) = \frac{2}{\sqrt{\pi}}\left[\int_0^x e^{-t^2}dt + \int_x^\infty e^{-t^2}dt\right]$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = 1$$

$$\Rightarrow \boxed{erf_c(x) = 1 - erf(x)} \qquad ...(3.7.3)$$

3.8. Summary

1. Properties of Beta and Gamma functions, recursion formula of $\Gamma(n)$ and their relation have been discussed.
2. Differentiation of indefinite and definite integrals and error function have been discussed.

3.9. Review Questions and Answer :

(1) Define the gamma function $\Gamma(n)$. Evaluate $\Gamma\left(-\frac{5}{2}\right)$ using $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Ans. See Text Book.

2nd part : $\Gamma(n + 1) = n\,\Gamma(n) \Rightarrow \Gamma\left(-\frac{5}{2}\right) = \dfrac{\Gamma\left(1-\frac{5}{2}\right)}{-\frac{5}{2}} = \dfrac{\Gamma\left(-\frac{3}{2}\right)}{-\frac{5}{2}}$

or, $$\Gamma\left(-\frac{5}{2}\right) = \frac{\Gamma\left(1-\frac{3}{2}\right)}{-\frac{5}{2}\times-\frac{3}{2}} = \frac{\Gamma\left(-\frac{1}{2}\right)}{-\frac{15}{4}} = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{\frac{15}{4}\times\left(-\frac{1}{2}\right)}$$

$$= \frac{\sqrt{\pi}}{-\frac{15}{8}} = -\frac{8\sqrt{\pi}}{15}.$$

2. The gamma function may be defined as

$\Gamma(n) = \int_0^{\infty} e^{x} x^{n-1} dx.$ Is there any restriction on the values of n? Why? "The gamma function $\Gamma(n)$ has finite values for all negative values of n" — is the statement true?

Or, Define gamma function $\Gamma(n)$ where n is not necessarily an integer and hence find recurrence relation $\Gamma(n + 1) = n\,\Gamma(n)$. Find the value of $\Gamma(1)$ from your definition and show that when n is positive integer $\Gamma(n) = (n - 1)$. Does the definition of $\Gamma(n)$ hold for $n = 0, -1, -2, -3,$ .... etc?

Ans. From the above definition. We obtain

$\Gamma(1) = \int_0^{\infty} e^{-x} dx = 1$ and we have from the defination.

$$\Gamma(n+1) = \int_0^{\infty} e^{-x} x^n dx = -x^n e^{-x}\Big]_0^{\alpha} + n\int e^{-x} x^{n-1} dx$$

$$= 0 + \text{n } \Gamma(\text{n})$$

$$\therefore \quad \Gamma(n+1) = n\Gamma(n) \Rightarrow \Gamma(n) = \frac{\Gamma(n+1)}{n}$$

When $n \to 0$, $\Gamma(n) \to \infty$ i.e., $\Gamma(0) \to \infty$

Hence, $\Gamma(-1) = \frac{\Gamma(0)}{-1} \to \alpha, \Gamma(-2) = \frac{\Gamma(-1)}{-2} \to \alpha,$ and so on.

The above argument shows that $\Gamma(n)$ has finite values when n is only possitive integer and when $n = 0, -1, -2,$ ..... i.e., zero or negative integer, then the definition of $\Gamma(n)$ does not hold. This is the restriction. $\Gamma\left(-\frac{3}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}\right)}{\left(-\frac{3}{2}\right)} = \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{3}{2} \times -\frac{1}{2}} = \frac{4\sqrt{\pi}}{3}$ shows that n can be negative odd half integer. But it only can be derived from the recurrence relation not from the definition of $\Gamma(n)$. This is also the restriction $\Gamma(n)$ converges only when n is positive integer.

3. Show that $\Gamma(n)=\int_0^1\left[ln\left(\frac{1}{y}\right)\right]^{n-1}dy,\ n>0$

Solution : We have, $\Gamma(n)=\int_0^\infty e^{-t}t^{n-1}dt$

Let us put, $e^{-t} = y \Rightarrow dy = -e^{-t}\, dt = y\, dt$

Now, $\Gamma(n)=\int_1^0 y\left[\ln\frac{1}{y}\right]^{n-1}\left(-\frac{dy}{y}\right)$

or, $\Gamma(n)=\int_0^1\left[\ln\left(\frac{1}{y}\right)\right]^{n-1}dy$

4. Establish the relation

$$\beta(m,n)=\frac{1}{2^{2m-1}}\beta\left(\frac{1}{2},m\right)$$

We have, $\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$

or, $\beta(m,m)=\int_0^1\left(x-x^2\right)^{m-1}dx$

$$= \int_0^1 \left[\frac{1}{4} - \left(\frac{1}{2} - x\right)^2\right]^{m-1} dx$$

$$= 2\int_0^{\frac{1}{2}} \left[\frac{1}{4} - \left(\frac{1}{2} - x\right)^2\right]^{m-1} dx$$

[As integral is symmetrical about $x = \frac{1}{2}$]

Now put, $\frac{1}{2} - x = \frac{1}{2} t^{\frac{1}{2}}$

$$dx = -\frac{1}{4} t^{-\frac{1}{2}} dt$$

$$\beta(m,m) = 2\int_1^0 \left(\frac{1}{4} - \frac{t}{4}\right)^{m-1} \left(-\frac{1}{4}\right) t^{-\frac{1}{2}} dt$$

$$= \int_0^1 \frac{(1-t)^{m-1}}{2^{2m-2}.2} t^{-\frac{1}{2}} dt$$

$$= \frac{1}{2^{2m-1}} \int_0^1 t^{-\frac{1}{2}} (1-t)^{m-1} dt$$

$$\boxed{\beta(m,n) = \frac{1}{2^{2m-1}} \beta\left(\frac{1}{2}, m\right)}$$ **(Proved)**

5. Express the following integrals in terms of gamma functions.

(i) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^4}}$, (ii) $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta$ (iii) $\displaystyle\int_0^1 x^4\left[\ln\left(\frac{1}{x}\right)\right]^3 dx$

Solution (i) $\displaystyle I=\int_0^1 \frac{dx}{\sqrt{1-x^4}}$

Let us put $x^2 = \sin\theta$

$$\Rightarrow\ dx=\frac{1}{2}\sin\theta^{-\frac{1}{2}}\cos\theta d\theta$$

$$\therefore\quad I=\int_0^{\frac{\pi}{2}}\sin^{-\frac{1}{2}}\theta d\theta$$

$$=\frac{1}{2}\cdot 2\int_0^{\frac{\pi}{2}}\sin^{\left(2\cdot\frac{1}{4}-1\right)}\theta\cos^{\left(2\cdot\frac{1}{2}-1\right)}\theta$$

$$=\ \frac{1}{2}\beta\left(\frac{1}{4},\frac{1}{2}\right)=\frac{1}{2}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}\ =\ \frac{\sqrt{\pi}}{4}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

(ii) $$\int_0^{\frac{\pi}{2}}\tan^{\frac{1}{2}}\theta\, d\theta=\frac{1}{2}\cdot 2\int_0^{\frac{\pi}{2}}\sin^{\left(2\cdot\frac{3}{4}-1\right)}\theta\cos^{\left(2\cdot\frac{1}{4}-1\right)}\theta\ d\theta$$

$$= \frac{1}{2}\beta\left(\frac{3}{4},\frac{1}{4}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)} = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)$$

(iii) $\int_0^1 x^4\left(\ln\frac{1}{x}\right)^3 dx$, put $x = e^{-\frac{t}{5}} \Rightarrow \ln\left(\frac{1}{x}\right) = \frac{t}{5}$ and $dx = -\frac{1}{5}e^{-\frac{t}{5}}dt$

$$= \int_\infty^0 e^{-\frac{4t}{5}}\frac{t^3}{125}\left(-\frac{1}{5}\right)e^{-\frac{t}{5}}dt$$

$$= \frac{1}{625}\int_0^\infty e^{-t}t^3 dt = \frac{1}{625}\,\Gamma(4) = \frac{6}{625}$$

6. Define the error function *erf* (x). Show that if *erf* (0) = 0. What is *erf* (∝)? Draw the graph of the error function.

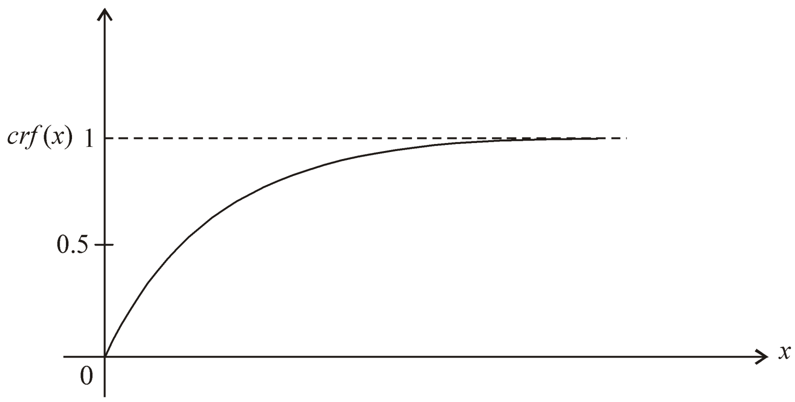
Ans. From (3.7.1), we obtain

$$erf\,(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

Now, *erf* (0) = 0 [from the definition]

$$erf\,(\infty) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\,dt = \frac{2}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2} = 1$$ [already discussed in article 3.3.1]

The graph from 0 to ∝ is shown below :



7. Show that *erf* (*x*) + *erf* (– *x*) = 0

Ans. $$erf\left(x\right)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt=-\frac{2}{\sqrt{\pi}}\int_0^{-x} e^{-t^2}dt=-erf\left(-x\right)$$

or, $erf\ (x) + erf\ (-x) = 0$ **(Proved)**

8. Establish the relation between the Beta function and the gamma function.

Ans. See article no. 3.4.

9. Show that $\frac{d}{dx}\left[erf\left(ax\right)\right]=\frac{2a}{\sqrt{\pi}}e^{-a^2x^2}$

Ans. We have,

$$erf\left(u\right)=\frac{2}{\sqrt{\pi}}\int_0^u e^{-t^2}dt$$

Differentiating both sides, w.r. t. u we get

$$\frac{d}{du}\left[erf\left(u\right)\right]=\frac{2}{\sqrt{\pi}}e^{-u^2}$$

$u = ax$, then we can write,

$$\frac{df}{dx}=\frac{df}{du}\frac{du}{dx}=a\frac{df}{du}$$

or, $\dfrac{df}{du} = \dfrac{1}{a}\dfrac{df}{dx}$

or, $\dfrac{d}{du}\left[erf(u)\right] = \dfrac{1}{a}\dfrac{d}{dx}\left[erf(x)\right]$

Hence we can write, $\dfrac{1}{a}\dfrac{d}{dx}\left[erf(x)\right] = \dfrac{2}{\sqrt{\pi}}e^{-a^2x^2}$

or, $\dfrac{d}{dx}\left[erf(x)\right] = \dfrac{2a}{\sqrt{\pi}}e^{-a^2x^2}$ **(Proved)**

10. Show that $\int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\, d\theta \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\sin\theta}} = \pi$

Ans. $I = \int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\, d\theta \int_0^{\frac{\pi}{2}} \dfrac{d\theta}{\sqrt{\sin\theta}}$

$$= \frac{1}{4}\left[\left\{2\int_0^{\frac{\pi}{2}} \sin^{\left(2\cdot\frac{3}{4}-1\right)}\theta \cos^{\left(2\cdot\frac{1}{2}-1\right)}\theta\, d\theta\right\}\left\{2\int_0^{\frac{\pi}{2}} \sin^{\left(2\cdot\frac{1}{4}-1\right)}\theta \cos^{\left(2\cdot\frac{1}{2}-1\right)}\theta\, d\theta\right\}\right]$$

$$= \frac{1}{4}\beta\left(\frac{3}{4},\frac{1}{2}\right)\beta\left(\frac{1}{4},\frac{1}{2}\right) = \frac{1}{4}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)}\cdot\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{1}{4}\cdot\frac{\sqrt{\pi}\cdot\sqrt{\pi}\,\Gamma\left(\frac{1}{4}\right)}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} = \pi$$ **(Proved)**

11. Show that $\beta\,(m, n)\,\beta\,(m + n, l) = \beta\,(n, l)\,\beta\,(n + l, m)$

Ans. $\beta\,(m, n)\,\beta\,(m + n, l) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \times \dfrac{\Gamma(m+n)\Gamma(l)}{\Gamma(m+n+l)}$

$$= \frac{\Gamma(m)\Gamma(n)\Gamma(l)}{\Gamma(m+n+l)} = \frac{\Gamma(n)\Gamma(l)}{\Gamma(n+l)} \times \frac{\Gamma(n+l)\Gamma(m)}{\Gamma(n+l+m)}$$

$= \beta\,(n, l)\,\beta\,(n + l, m)$ **(Proved)**

12. Show that $\displaystyle\int_0^1 \frac{dx}{\left(1-x^6\right)^{\frac{1}{6}}} = \frac{\pi}{3}$

Solution : L.H.S. = $\displaystyle\int_0^1 \frac{dx}{\left(1-x^6\right)^{\frac{1}{6}}}$

Put $x^6 = z$, or, $6x^5 dx = dz$ or, $dx = \dfrac{dz}{z^{\frac{5}{6}}}$

$$= \frac{1}{6}\int_0^1 \frac{dz}{z^{5/6}(1-z)^{1/6}}$$

$$= \frac{1}{6}\int_0^1 z^{-\frac{5}{6}}\left(1-z\right)^{-\frac{1}{6}} dz$$

$$= \frac{1}{6}\int_0^1 z^{\frac{1}{6}-1}\left(1-z\right)^{\frac{5}{6}-1} dz$$

$$= \frac{1}{6}\beta\left(\frac{1}{6},\frac{5}{6}\right)$$

$$= \frac{1}{6}\frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma(1)}$$

$$= \frac{1}{6}\Gamma\left(\frac{1}{6}\right)\Gamma\left(1-\frac{1}{6}\right)$$

$$= \frac{1}{6}\cdot\frac{\pi}{\sin\frac{\pi}{6}} = \frac{\pi}{3} \quad [\because \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}]$$

13. Evaluate the integral $\int_0^\infty 3^{-4z^2}dz$

Solution : $\int_0^\infty 3^{-4z^2}dz$

$$= \int_0^\infty \left(e^{\ln 3}\right)^{-4z^2} = \int_0^\infty e^{-(4\ln 3)z^2}dz$$

Put $(4 \ln 3)z^2 = x$ or, $dz = \frac{x^{-\frac{1}{2}}dx}{2\sqrt{4\ln 3}}$

$$= \frac{1}{2\sqrt{4\ln 3}}\int_0^\infty e^{-x}x^{-\frac{1}{2}}dx$$

$$= \frac{1}{2\sqrt{4\ln 3}}\cdot\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2\sqrt{4\ln 3}} = \frac{1}{4}\sqrt{\frac{\pi}{\ln 3}}$$

14. Evaluate : $\displaystyle\int_0^1 \frac{dx}{\left(1-x^n\right)^{\frac{1}{n}}}$

Solution : Put $x^n = \sin^2\theta$ or, $x = \sin^{\frac{2}{n}}\theta$

$$dx = \frac{2}{n}\sin^{\frac{2}{n}-1}\theta\cos\theta\, d\theta \qquad \therefore \int_0^1 \frac{dx}{\left(1-x^n\right)^{\frac{1}{n}}} = \int_0^{\frac{\pi}{2}} \frac{\frac{2}{n}\sin^{\frac{2}{n}-1}\theta\cos\theta\, d\theta}{\cos^{\frac{2}{n}}\theta}$$

$$= \frac{2}{n}\int_0^{\frac{\pi}{2}} \sin^{\left(\frac{2}{n}\right)-1}\theta\cos^{\left(1-\frac{2}{n}\right)}\theta\, d\theta$$

$$= \frac{2}{n}\cdot\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)}{2\Gamma(1)} \qquad \because \int_0^{\frac{\pi}{2}} \sin^p\theta\cos^q\theta\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)$$

$$= \frac{\pi}{n\sin\left(\frac{\pi}{n}\right)}$$

15. Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin^6\theta\, d\theta$

Solution : We have $\displaystyle\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\, d\theta = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}$

Given $\int_0^{\frac{\pi}{2}} \sin^6 \theta \, d\theta$

$\therefore$ $2m - 1 = 6$ or, $m = \frac{7}{2}$, and $2n - 1 = 0$ or, $n = \frac{1}{2}$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^6 \theta d\theta = \frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(4)} = \frac{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\pi}{2 \times 6} = \frac{5\pi}{32}$$

16. Show that $\boldsymbol{erf(x) = \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \ldots\right)}$

Solution. We have $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2} dy$

When x is small we can write

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x \left[1 - \frac{y^2}{1!} + \frac{y^4}{2!} - \frac{y^6}{3!} + \frac{y^8}{4!} - \ldots\right] dy$$

$$= \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3.1!} + \frac{x^5}{5.2!} - \frac{x^7}{7.3!} + \ldots\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \ldots\right]$$

17. Compute *erf* (0.3).

Solution. We have

$$erf\,(x) = \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \ldots\right]$$

Putting $x = 0.3$

$$erf\,(0.3) = \frac{2}{\sqrt{\pi}}\left[0.3 - \frac{(0.3)^3}{3} + \frac{(0.3)^5}{10} - \frac{(0.3)^7}{42} + \ldots\right]$$

$= 1.128\ [0.3 - 0.009 + 0.000243 + \ldots\ldots\ldots\ldots]$

$= 0.3286.$

[1]Unit–4 ❑ Theory of Errors

Structure

4.0 Objectives

When you go through this unit, you may be able to learn

1. Definition, propagation and normal law of errors.
2. Statistical methods in error analysis.

4.1 Introduction

The process of evaluating uncertainty regarding the measurement of any experimental results is called error analysis. The uncertainty of a single measurement of any kind of experiment is limited by the precision and accuracy of the measurement. Precision is a measurement without the reference of true value. It is based on the degree of consistency and agreement of independent measurements of the same quantity. Precision determines the reproducibility of the measurements. Where as accuracy is the closeness of agreement between the measured value and the true value. Hence we can say Error is nothing but the inaccuracy of the results which we want to measure.

4.2 Definition of Errors :

The difference between the value, which we want to obtain and what we already measured, is called error. Due to wrong calibration of the measuring instrument, a constant error can take place during measurement. This type of error is called instrumental error, which can be adjusted only by the reference data. Except this instrumental error, there are different types of errors can arise in a measurement. Generally the errors are classified by two types : 1. Random errors and 2. Systematic errors (includes instrumental errors).

1. Random Errors : The errors occuring due to unknown reasons are called random errors. It can arise due to alteration of experimental conditions that are beyond the control ; examples are vibrations in the experiment, fluctuation of temperature and humidity, pressure etc. The detection of random errors are very difficult, because their effect on the experimental values is different for every repetition of the experiment. Instrumental resolution also be the cause of random erros, because all instruments have finite precision that limits the ability to resolve the small differences of the measurement. The alignment of eye with the pointer is one of the key reasons of random errors. Hence the statistical methods are used to obtain the random errors.

2. Systematic erros : The errors which are governed by some systematic rule is called systematic errors. For example, measuring a distance using the wrong end of a meter stick, incorrectly neglecting the effect of viscosity, air resistance, friction etc., that can provide a systematic shift of the measuring data. Some measuring devices require time to reach its equilibrium conditions. The most common example is taking temperature readings with a thermometer that is not thermally equilibrium with its environment.

4.3 Propagation of errors :

As discussed earlier, there are uncertainties in measurement which we call them errors (systematic or Random). Since all measurements have uncertainties associated with them, then question arrises, how can we determine the values of uncertainties?

If y is a function of x i.e.y = y(x), then uncertainty associated with x is $x \pm \Delta x$ and in this way y also.

As $y = y(x)$

$$\text{then } \Delta y = \frac{\partial y}{\partial x}\Delta x \qquad \text{... (4.3.1)}$$

Let us consider f is a function of independent variables x, y, z ...... i. e.

$$f = f\,(x,\ y,\ z\ ...) \qquad ...(4.3.2)$$

The expectation values and uncertainties associated with the independent variables are $< x >$, $< y >$, $< z >$ .... and Δx, Δy, Δz ..... respectively. Hence the expectation value of f can be written as

$$< f > = f\,(< x >,\ < y >,\ < z > .....) \qquad ...(4.3.3)$$

and uncertainty of f is given by

$$|\Delta f| = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z} + \Delta z + ...$$

$$\text{or, } |\Delta f| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 \Delta x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \Delta y^2 + ...} \qquad ...(4.3.4)$$

Hence the calculated quantity f is given by

$$f = < f > \pm \Delta f \qquad ...(4.3.5)$$

There are different ways by which we can calculate the values of f. The ways are discussed below :

1. Addition or Subtraction :

Suppose two quantities x and y are added i.e.,

$$f\,(x,\ y) = x + y$$

From equation (4.3.3) we get $< f > = < x > + < y >$

and from equation (4.3.4) we get $|\Delta f| = \sqrt{\Delta x^2 + \Delta y^2}$

Now for susbtraction $f\,(x,\ y) = x - y$

$$\Rightarrow < f > = < x > - < y >$$

and $|\Delta f| = \sqrt{\Delta x^2 + \Delta y^2}$ [From equation (4.3.4)]

Hence for any combination of addition and subtraction the uncertainty for measurements of any values gives the same value i.e.,

$$|\Delta f| = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2 + \dots} \qquad \dots(4.3.6)$$

2. Multiplication or Division :

Suppose two quantities are multiplied i.e.,

$$f = xy$$

$$\Rightarrow < f > = < x > < y >$$

where as $|\Delta f| = \sqrt{y^2 \Delta x^2 + x^2 \Delta y^2}$ [From equation (4.3.4)]

$$\text{or, } |\Delta f| = \sqrt{\left(x^2 y^2\right)\left[\frac{\Delta x^2}{x^2} + \frac{\Delta y^2}{y^2}\right]}$$

$$= f\sqrt{\frac{\Delta x^2}{x^2} + \frac{\Delta y^2}{y^2}}$$

For division say $f = \dfrac{x}{y}$

$$\Rightarrow < f > = \frac{< x >}{< y >}$$

and $|\Delta f| = \sqrt{\dfrac{\Delta x^2}{y^2} + \dfrac{x^2}{y^4} \Delta y^2}$ [From equation (4.3.4)]

$$= \sqrt{\frac{x^2}{y^2}\left(\frac{\Delta x^2}{x^2} + \frac{\Delta y^2}{y^2}\right)}$$

$$= f\sqrt{\frac{\Delta x^2}{x^2} + \frac{\Delta y^2}{y^2}}$$

Hence for division or multiplication, the uncertainty of any measurement gives the same result and the values are

$$|\Delta f| = f\sqrt{\frac{\Delta x^2}{x^2} + \frac{\Delta y^2}{y^2} + \frac{\Delta z^2}{z^2} + ...} \qquad ...(4.3.7)$$

3. Raising to a power :

Suppose the function f is defined as

$$f = x^n \, y^m$$

$$\Rightarrow \log f = n \log x + m \log y$$

or, $< \log(f) > = n < \log(x) > + m < \log(y) >$

and by differentiating both sides we get,

$$\frac{\Delta f}{f} = n\frac{\Delta x}{x} + m\frac{\Delta y}{y}$$

or, $$\left|\frac{\Delta f}{f}\right| = \sqrt{n^2\left(\frac{\Delta x}{x}\right)^2 + m^2\left(\frac{\Delta y}{y}\right)^2}$$

From equation (4.3.4) we obtain

$$|\Delta f| = \sqrt{\left(nx^{n-1}y^m\right)^2 \Delta x^2 + \left(my^{m-1}x^n\right)^2 \Delta y^2}$$

$$= \sqrt{n^2\left(x^n y^m\right)^2 \frac{\Delta x^2}{x^2} + m^2\left(x^n y^m\right)^2 \frac{\Delta y^2}{y^2}}$$

$$= f\sqrt{n^2\frac{\Delta x^2}{x^2} + m^2\frac{\Delta y^2}{y^2}}$$

In general, for multivariables

$$|\Delta f| = f\sqrt{n^2\frac{\Delta x^2}{x^2} + m^2\frac{\Delta y^2}{y^2} + ...} \qquad ...(4.3.8)$$

For all these equations Δx, Δy, Δz ..... represents the absolute error and $\frac{\Delta x}{x}, \frac{\Delta y}{y}, \frac{\Delta z}{z}$ ..... represents relative error. So the percentage error is

$$\boxed{\text{Error\%} = \frac{\Delta f}{f} \times 100}.$$

4.4 Normal law of Errors :

A series of errors are obeying the Normal law. The accidental errors associated with an extended series of observations, called normal law of errors. The exponential law for the distribution of acccidental errors of observation, discovered by Gauss has been a mathematical classic for over a century. If a measurement is subject to many small sources of random error and negligible systematic error, the limiting distribution will have the form of the smooth bell-shaped curved as shown in Fig. 4.1.

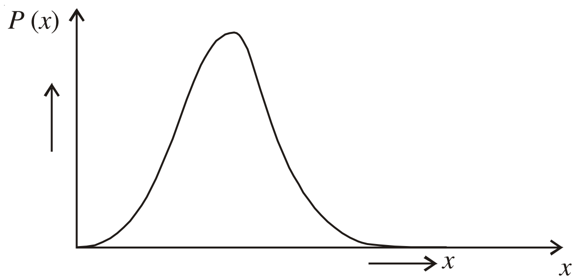


Fig. 4.1

This distribution curve is very often, called Gaussian distribution curve. This curve will be centered on the true value of the measured quantity. In general, this limiting distribution defines a function $P(x)$. From the symmetry of the bell-shaped curve, we can say $P(x)$ is centered on the average value of x. We know the mean or expectation value for finite number of measurements are given by

$$< x >= \sum_{n=1}^{\infty} \frac{x_1 + x_2 + ... + x_n + ...}{N.} = \mu \text{ (say)}$$

$$\therefore \quad \mu = \underset{N \to \infty}{Lt} \frac{1}{N} \sum_{i=1}^{N} x_i \qquad ...(4.4.1)$$

Here μ is the true value in the absence of systematic errors. The values of μ is confined with μ ± σ, where σ is the standard deviation, defined by

$$\sigma = \sqrt{\underset{N \to \infty}{Lt} \frac{1}{N} \sum_{i=1}^{N} (x_i - \mu)^2} \qquad ...(4.4.2)$$

The standard deviation, σ characterizes the average uncertainty in each of the measurements x_i. Hence $P(x)$ is directly related with μ and σ. Gauss showed that, for randomly distributed errors, the distribution function is given by

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} \qquad ...(4.4.3)$$

The measurements whose limiting distribution N given by the Gauss function are said to be normally distributed.

The significance of this function is shown in Fig 4.2. The fraction of measurements that fall in any small interval x to $x + dx$ is equal to the area $P(x)\, dx$ of the strip as shown in Fig 4.2. The total probability of our measurement falling anywhere between $-\infty$ to $+\infty$ must be unity i.e.

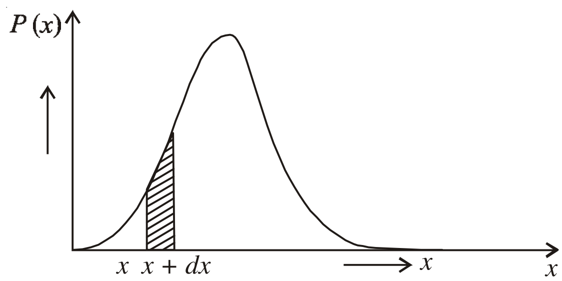


Fig. 4.2

$$\int_{-\infty}^{\infty} P(x)\,dx = 1 . \qquad \text{...(4.4.4)}$$

Now for finite number of measurements of x_i with the results : x_1, x_2 ..... x_N, the probability of finding a value of x within the interval

x_1, $x_1 + dx_1$; x_2, $x_2 + dx_2$, ... x_N, $x_N + dx_N$ are given by

$$P(x_1) = \frac{1}{\sigma\sqrt{2\pi}} \exp.\left\{-\frac{1}{2}\left(\frac{x_1-\mu}{\sigma}\right)^2\right\}$$

$$P(x_2) = \frac{1}{\sigma\sqrt{2\pi}} \exp.\left\{-\frac{1}{2}\left(\frac{x_2-\mu}{\sigma}\right)^2\right\}$$

....

$$P(x_N) = \frac{1}{\sigma\sqrt{2\pi}} \exp.\left\{-\frac{1}{2}\left(\frac{x_N-\mu}{\sigma}\right)^2\right\}$$

$\therefore$ P (x_1, x_2, ... x_N)

$= P$ (x_1) $P(x_2)$ ..... $P(x_N)$

$$= \frac{1}{\sigma\sqrt{2\pi}} \exp.\left\{-\sum_{i=1}^{N} (x_i-\mu)^2 \big/ 2\sigma\right\} \qquad \text{...(4.4.5)}$$

In equation (4.3.5), the numbers μ and σ are unknown, and we want to find the best estimates for μ and σ based on the given observations x_1, x_2 ..... x_N. From equation (4.4.5), we can say that probability is maximum when

$$\sum_{i=1}^{N} (x_i-\mu)^2 \big/ (2\sigma) \text{ is minimum.}$$

i.e., $\sum_{i=1}^{N} (x_i-\mu) = 0$

or, $\sum_{i=1}^{N} x_i - N\mu = 0$

$$\therefore \quad \mu = \frac{1}{N}\sum_{i=1}^{N} x_i$$

Proceeding in a same manner, the best estimation of

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2}$$

$$= \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \mu)^2} \qquad ...(4.4.6).$$

Hence N is replaced by $(N - 1)$, because the calculatIon of μ has used up one independent piece of information.

One of the most popular method by which one can measure the near accurate value of any experiment is <u>least square fitting</u>. Most common types experiment involves the measurement of several values of two different variables to investigate the mathematical relation between two variables.

Let us consider the general case of two variables x and y are related with a linear relation

$y = A + Bx.$ *...(4.4.7).,* where A and B are constants.

Now if we want to measure N different values $\{y_1\ y_2, ...\ y_N\}$ corresponding to N values of $\{x_1, x_2, ...\ x_N\}$, then each points (x_i, y_i) would lie exactly on the line given in equation (4.4.7).

Now we want to find out the values A and B, that give the best straight line fit to the measured data.

If we know the constants A and B, then for any gieven values of x_i, we can calculate the true value of the corresponding y_i :

$$y_i = A + B\,x_i \qquad ...(4.4.8)$$

From our assumptions, the measurement of y_i is governed by a normal distribution centered on this true value, with width parameter σ_y (say). Therefore, the probability of obtaining the observed value y_i is

$$P(y_i) = P(y_1) \ldots.. P(y_N)$$

or, $$P(y_i) = \frac{1}{\sigma_y\sqrt{2\pi}}\exp.\left\{-\frac{1}{2\sigma_y^{\,2}}(y_1-\bar{y})^2 \ldots.. -\frac{1}{2\sigma_y^{\,2}}(y_N-\bar{y})^2\right\}$$

$$= \frac{1}{\sigma_y\sqrt{2\pi}}\exp.\left\{-\frac{1}{2\sigma_y^{\,2}}\left(\sum_{i=1}^{N} y_i - N\bar{y}\right)^2\right\}$$

$$= \frac{1}{\sigma_y\sqrt{2\pi}}\exp.\left\{-\frac{1}{2\sigma_y^{\,2}}\left(\sum_{i=1}^{N} y_i - A - Bx_i\right)^2\right\}$$

$$= \frac{1}{\sigma_y\sqrt{2\pi}}\exp.-\frac{X^2}{2\sigma y^2} \qquad \ldots(4.4.9), \text{ where}$$

$$X^2 = \sum_{i=1}^{N}(y_i - A - Bx_i)^2$$

The probability is maximum when X^2 is minimum in equation (4.4.9). This is why this method is called least square fitting.

Now, $$\frac{d\left(X^2\right)}{dA} = -2\sum_{i=1}^{N}(y_i - A - Bx_i) = 0 \qquad \ldots(4.4.10)$$

and $$\frac{d\left(X^2\right)}{dB} = -\sum_{i=1}^{N} 2x_i(y_i - A - Bx_i) = 0 \qquad \ldots(4.4.11)$$

Equation (4.3.10) gives $AN + B\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} y_i$ ...(4.4.12)

and equation (4.3.11) gives $A\sum_{i=1}^{N} x_i + B\sum_{i=1}^{N} x_i^{\,2} = \sum_{i=1}^{N} x_i y_i$ ...(4.4.13)

Equation (4.4.12) and (4.4.13) are called normal equations.

Equating these two equations we obtain

$$A = \frac{\sum_{i=1}^{N} x_i^2 \sum_{i=1}^{N} y_i - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} x_i y_i}{N\sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2}$$

$$and\ B = \frac{\left. N\sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i \right\}}{N\sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2} \qquad ...(4.4.14)$$

Putting the values of equation (4.4.14) in equation (4.4.8), one can easily solve for the best least squre estimates of A and B.

4.5 Statistical methods in error analysis :

The statistical methods help us to reduce the random errors. For a set of data containing N elements or measurements, given by $\{P_1, P_2, P_3 \ ..... \ P_N\}$, the average or expectation value

$$\langle P \rangle = \frac{P_1 + P_2 + P_3 + ... + P_N}{N} = \frac{1}{N}\sum_{i=1}^{N} P_i \qquad ...(4.5.1)$$

The values of $\langle P \rangle$ is sometimes referred to as the best estimate of the actual value. The data P_i are dispersed arround the mean or average. The measurement of this dispersion is called the standard deviation and is given by

$$\Delta P = \sqrt{\frac{\left(P_1^2 + P_2^2 + ... + P_N^2\right)}{N} - \langle P\rangle^2}$$

$$= \sqrt{\frac{\left(P_1^2 + P_2^2 + ... + P_N^2\right) - N\langle P\rangle^2}{N}}$$

$$= \sqrt{\frac{1}{N}\left[\sum_{i=1}^{N} P_i^2 - N\langle P\rangle^2\right]} \qquad ...(4.5.2)$$

Here ΔP is defined with a factor $(N - 1)$ rather than N. When N is very large number, then the values of ΔP is not diferred as $N - 1 \approx N$, but when N is a small number, then deviation occurs. The question arises why we are taking $(N - 1)$, rather than N? This is simply because, one information we are taking independent or reference from the set of values. Hence equation (4.2.2) becomes

$$\Delta P = \sqrt{\frac{1}{N-1}\left[\sum_{i=1}^{N} P_i^2 - N\langle P\rangle^2\right]} \qquad ...(4.5.3)$$

When the standard deviation is small, then it is close to the mean value and they fall within the interval between $(P - \Delta P)$ and $(P + \Delta P)$. If the systematic errors are reduced, the random errors will dominate and it is impossible to get the accurate result. There is a way to reduce the random errors. Equation (4.5.3) shows that if N is large, then ΔP is small. If we take the more and more data points as much as posssible in an experiment, then it is possible to reduce the random erros. This type of survey or data analysis is called statistical approach.

4.5.1 Standard error and probable error :

1. The standard error is a measure of the variability of a statistic. It is an estimate of the standard deviation of a sample distribution. The procedure of standard error calculation is given below.

(a) First you have to calculate the mean.

(b) Calculate the deviation of all samples from the mean i.e., mean minus the individual measurement.

(c) Then square each deviation from mean, so that positive or negative values will give you only positive value.
(d) Sum the standard deviation.
(e) Divided that sum of standard deviation by the number of measurements minus one.
(f) To get the standard deviation, square root the values of (e).
(g) Divide the standard deviation by the square root of the sample size(n). In this way we obtain the standard error.
(h) Add and subtract the standard error from the mean and record the number.

2. **Probable error :** Probable error defines the half-range of an interval about a central point for the distribution, such that half of the values from the distribution will lie, within the interval and half outside. The probable error is expressed as equal to 0.6745 times the standard deviation.
The probable error is defined as

Probable error $= 0.6745 \times \frac{l - r^2}{\sqrt{N}}$ (4.5.4), where N is the total number of observations and r is the correlation co-efficient of pairs of observations for any random sample. Correlation co-efficients is the covariance of the two variables divided by the product of their standard deviations.

If there is a pair of random variables (X, Y), then $r = \frac{\mathrm{Cov}(X,Y)}{\sigma_x \sigma_y}$, [where covariance or variance is denoted by Cov (X,Y), and σ_X, σ_Y are dicussed earlier]
If $X = X\ (x_1,\ x_2\ \ldots\ldots\ x_N)$
and $Y = Y\ (y_1,\ y_2,\ \ldots\ldots\ y_N)$

$$Cov(X,Y) = \sum_{i=1}^{N} < (x_i - \mu)^2 > < (y_i - \mu)^2 >$$

$$\left.\begin{aligned} &\text{and} \quad \sigma_x = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i-\mu)^2} \\ &\text{and} \quad \sigma_y = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(y_i-\mu)^2} \end{aligned}\right\} \text{[From equation...4.5.3]}$$

The probable error is also defined as

Probable error = $0.6745\frac{1-\rho^2}{\sqrt{N}}$ $(1+1.086\rho^2 + 0.13\rho^4 + 0.002\rho^6)$, where ρ is the correlation of a population. This is also known as limit of the correlation of co-efficient.

4.5.2 Conditions to find probable error :

1. The distribution of the data must have the bell-shaped curve i.e., normal distribution curve.
2. It is important to calculate probable error measuring the statistics from the sample only.
3. The samples are taken in such a manner so that they must remain independent of each other's value.

4.5.3 Advantages of standard error :

1. Standard errors help us to reduce the sample errors as well as the measurement errors.
2. The standard error of any mean tells about the accuracy of the estimate clearly.

4.6 Percentage Error :

The deviation is the measure of the precision of an experiment i.e., smaller the value of deviation means estimation tends to the accurate value. Generally in experimental physics it is need to account the accuracy i.e., how the difference is between the experimental values and established values (Theoritical values). The percentage error is represented by

$$\text{Error\%} = \frac{\text{Experimental value} - \text{Theoritical value}}{\text{Theoritical value}} \times 100\% \qquad ...(4.6.1)$$

If we say experimental value = S_1
and theoritical value = S_2, then deviation is

$$\Delta S = |S_1 - S_2| \qquad ...(4.6.2)$$

Most of the case the true experimental value is unknown. In this case, it is often useful to take the results from two different methods, so that a difference can be obtain and for our assumption we can take the mean of this two values to get near to the accurate value, then the percentage error may be represented by

$$\text{Errors\%} = \frac{|S_1 - S_2|}{\left|\frac{S_1 + S_2}{2}\right|} \times 100\% \qquad ...(4.6.3)$$

4.7 Summary

1. Different types of errors, propagation of errors and normal law of errors have been discussed.
2. Statistical methods in error analysis such as standard error and probable error have been discussed.
3. Conditions to find probable error and advantages of standard error and percentage error calculation have been discussed.

4.8 Review Questions and Answer :

1. A thermometer reads 190ºC, when the actual temperature is 195ºC. Find the percentage error in the reading.

Ans. Percentage error is given by

$$\text{Error\%} = \frac{|195 - 190|}{195} \times 100\% = 2.564\%$$

2. Distinguish between Random error and systematic error. Write down the two possible sources of random error and systematic error.

Ans. See article no. (4.2).

3. The most probable value of a set of dispersed data is arithematic mean. Justify the statement.

Ans. Let us take a set of experimental values [20, 20.5, 23, 23.5, 23.7, 24, 24.5, 25] for a single experiment. All the data are in the range of 20 to 25.The experiment was performed for eight (8) times and most probable value is around 23.

The mean value is given by

$$\mu = \frac{1}{N}\sum_{i=1}^{N} x_i$$

$$= \frac{1}{8}\ [20 + 20.5 + 23 + 23.5 + 23.7 + 24 + 24.5 + 25]$$

$$= 23.025$$

In this simple manner we can easily justify the above statement.

4. What are the different sources of errors?

Ans. (a) Systematic error. (see article no. (4.2))

(b) Random error. (see article no. (4.2))

(c) Least count error : Least count error is associated with the resolution of the instrument.

(d) Constant error

(e) Errors due to external factors : Discussed in systematic error.

5. The resistance value at a temperature t of a wire R_t is given by the relation $R_t = R_0\,(1 + \alpha\, t)$, where R_0 is the resistance at 0ºC and α is the temperature co-efficient of resistance. The resistance values of the metal wire at different temperature is tabulated below. Obtain the values of R_0 and α using least square straight line fitting.

Temperature (ºC) (t)	20	40	60	80	100
Resistance (Ω) (R_t)	107.5	117	129	138	145.5

Ans. The given equation is $R_t = R_0 \alpha t + R_0$ ....(1)

$$[y = \beta x + A]$$

From equations (4.4.14), we obtain

$$A = R_0 = \frac{\sum_{i=1}^{N} x_i^2 \sum_{i=1}^{N} y_i - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} x_i y_i}{N\sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2}$$

$$B = R_0\alpha = \frac{N\sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i}{N\sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2}$$

Here, $\sum_{i=1}^{5} x_i^2 = 22000 \Rightarrow 5\sum_{i=1}^{5} x_i^2 = 110000$

$$\sum_{i=1}^{5} x_i = 300 \Rightarrow \left(\sum_{i=1}^{5} x_i\right)^2 = 90000$$

$$\sum_{i=1}^{5} y_i = 637$$

$$\sum_{i=1}^{5} x_i y_i = 40160$$

$$\therefore\ R_0 = \frac{637 \times 22000 - 300 \times 40160}{110000 - 90000} = 9\dot{8}\cdot 3$$

and $R_0\alpha = \dfrac{5\times 40160 - 300\times 637}{110000 - 90000} = 0\cdot\dot{4}85$

$\therefore \quad \alpha = \dfrac{0.485}{98.3} = 0.0049$

6. In an experiment, the time period of an oscillating object in five successive measurements is found to be 0.52 *S*, 0.56 *S*, 0.57 *S*, 0.54 *S* and 0.59 *S*. The least count of the watch used for the measurement of time period is 0.01*S*. What is the percentage error in measurement of time period T.

Ans. The mean of the measured value is

$$\mu = \frac{1}{N}\sum_{i=1}^{N} x_i = \frac{1}{5}[0.52 + 0.56 + 0.57 + 0.54 + 0.59]S$$

$$= 0.56 \text{ S}$$

From equation (4.3.2) we obtain

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \mu)^2}$$

$$= \frac{1}{4}\left[(-0.04)^2 + 0^2 + (0.01)^2 + (0.02)^2 + (0.03)^3\right]$$

$$= 0.027\text{S}$$

Standard error = $\dfrac{\sigma}{\sqrt{N}} = \dfrac{0.027}{\sqrt{5}}$ = 0.012S

The percentage error = $\dfrac{\text{Standard error}}{\text{mean}} \times 100\%$

$$= \frac{0.012}{0.56} \times 100\%$$

$$= 2.14\%$$

7. How can we minimize errors?

Ans. (i) Using instruments of higher precision, improving experimental techniques, etc. we can reduce least count error. (ii) Repeating the observations, several times and taking the arithmetic mean of all the observation, the mean value would be very close to the true value of the measured quantity. (iii) Gross error can be minimized only if the observer is very careful and sincere in his approach.

8. What are the different ways of expressing an error?

Ans. (a) Absolute error

(b) Relative error

(c) Percentage error

9. What is called accuracy?

Ans. The accuracy of an instrument is a measure of how close the output reading of the instrument with the correct value.

10. A resistor is market with 470Ω, 10%. What will be the true value of the resistor?

Ans. A resistor is marked with 470Ω, 10% means the actual (true value) value of the resistor lies within (470 + 470 × 10%)Ω to (470 – 470 × 10%)Ω or 517Ω to 423Ω.

11. What is absolute error?

Ans. The absolute error of a measurement is the magnitude of the difference between the actual value (true value) and the value of the individual measurement. The actual or true value is taken by doing the arithmetic mean, because we are not sure about the actual value. If there are *n* number of readings [A_1, A_2, A_3,... A_n].

Arithnetic mean, $\bar{A} = \frac{1}{n}[A_1 + A_2 + ... + A_n]$

Hence for each measurement, corresponding errors can be represented as

$$|\Delta A_1| = |\bar{A} - A_1|$$

$$|\Delta A_2| = |\bar{A} - A_2|$$

....

$$|\Delta A_n| = |\bar{A} - A_n|$$

So, absolute error $|\Delta\bar{A}| = \frac{1}{n}[\Delta A_1 + \Delta A_2 + ... + \Delta A_n]$

12. The refractive index of water is found to have the values 1.29, 1.33, 1.34, 1.35, 1.32. Calculate the mean value, absolute error and percentage error.

Ans. Mean value $(\bar{\mu}) = \dfrac{1.29+1.33+1.34+1.35+1.32}{5} = 1.326$

Absolute error = $\dfrac{1}{5}\Big[|1.326-1.29| + |1.326-1.33| + |1.326-1.34| + |1.326-1.35|$

$+|1.326-1.32|\Big]$

$= 0.0168$

Percentage error $= \dfrac{\text{Absolute error}}{\text{Mean}} \times 100\%$

$= \dfrac{0{\cdot}0168}{1{\cdot}326} \times 100\%$

$= 1.27\%$

13. How error are propagated or combined?

Ans. See propagation error.

14. Find the probable error for the given correlation co-efficient 0.6 and the pairs of samples are 24.

Ans. The equation (4.5.4) gives

probable error $= 0.6745 \times \dfrac{1-r^2}{\sqrt{N}}$

$= 0.6745 \times \dfrac{1-(0.6)^2}{\sqrt{24}}$

$= 0.088$

Hence the error percentage of probable error = 0.088 × 100% = 8.8%.

15. Find out the standard error for a measurement of height distribution (152 cm, 155 cm, 160 cm and 162 cm).

Ans. Height distribution x_i = [152, 155, 160, 162]

Mean value $\mu = \frac{1}{4}[152+155+160+162]$ cm

$= 157.25$ cm

Standard deviation

$$\sigma = \sqrt{\frac{1}{N-1}\left(\sum_{i=1}^{N}(x_i-\mu)\right)^2} \quad \text{[From equation (4.4.2)]}$$

$$= \left[\frac{1}{3}(152-157.25)^2+(155-157.25)^2+(160-157.25)^2 +(162-157.25)^2\right]^{\frac{1}{2}} \text{ cm}$$

$= 4.57$ cm

Standard error $= \frac{\sigma}{\sqrt{N}} = \frac{4.57}{\sqrt{4}} = 2.29$ cm

Hence the true value = μ ± standard error

= (157.25 ± 2.29) cm

= 159.54 or 154.96 cm

Unit–5 ❑ Partial Differentiations

Structure :

5.0 Objectives

1. To solve a partial differential equation using method of separation of variables.
2. To illustrate the method to solve Laplace's equation and wave equation.

5.1 Introduction

Partial differential equation contain the rate of change of variables that are independent to each other. A function can define with multivariables e.g.,

$$u = f\,(x,\ y,\ z,\ t\ .....) \qquad ...(5.1.1)$$

A partial differential equation for the function u is an equation of the form

$$f\left(x, y, z, t.....; u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}.......\right) \qquad ...\ (5.1.2)$$

5.2 Differentials :

If u = $f(x)$, then for small change of Δx, a small change $\Delta\, u$ can be written as

$$\Delta u = \frac{du}{dx}\Delta x$$

or, $$du = \frac{du}{dx}dx \qquad ...(5.2.3)$$

In equation (5.2.3) dx and du are called the differentials of x and u respectively.

Now, if $u = f(x, y)$, then for small change of Δx and Δy (two indepedent variables), we can write

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \qquad ...(5.2.4)$$

Here, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ are called partial derivatives.

For multivariables, equation (5.2.4) can expressed as

$$du = \frac{\partial u}{\partial x_1} dx_1 + ... + \frac{\partial u}{\partial x_n} dx_n$$

$$= \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} dx_i \qquad ...(5.2.5)$$

Where $u = f(x_1, x_2 ....., x_n)$

5.3 Solution of partial differential Equations :

The most widely used method to solve a partial differential equation is, separation of variables. In this method, it is assured that, $u(x, y) = \phi(x)\psi(y)$, i.e., the function of many variables is a product of function, each of which is a function of single variables. Limitation of this method is, it is applicable only for homogeneous function that means the function $f(x, y)$ which can be expressed in the form $x^n \phi(x, y)$. In general, a function $f(x, y, z, t ...)$ is a homogeneous function of degree n, it is possible to express it in the form

$$x^n \phi\left(\frac{y}{x}, \frac{z}{x}, \frac{t}{x} .....\right).$$

Moreover it sometimes happens that coordinate system in which the separation is possible, is not suitable for applying the boundary conditions. We shall illustrate this method to solve the

(1) Laplace's equaton.

(2) Wave equation.

5.4 Laplace's equation :

I. The Laplace's equation in certesian coordinates is given by

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0 \qquad ...(5.4.1)$$

Let us consider,

$$\phi\ (x,\ y,\ z) = X\ (x)\ Y\ (y)\ Z\ (z) \qquad ...(5.4.2)$$

Now, substituting (5.4.2) in equation (5.4.1) we get,

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=0$$

Since, x, y, z are independent.

Let us write, $$\left.\begin{aligned}\frac{1}{X}\frac{d^2X}{dx^2}&=-k_1^2\\ \frac{1}{Y}\frac{d^2Y}{dy^2}&=-k_2^2\end{aligned}\right\} ......(5.4.3)$$

Hence $$\frac{1}{z}\frac{d^2Z}{dz^2}=\left(k_1^2+k_2^2\right)=k_3^2$$

From (5.4.3) we get,

$$X = Ae^{k_1x}+Be^{-k_1x}$$

$$Y = Ce^{k_2y}+De^{-k_2y} \qquad ...(5.4.4)$$

and $Z = E\cos k_3z + F\sin k_3z$

$$= E\cos\sqrt{\left(k_1{}^2 + k_2{}^2\right)}z + F\sin\sqrt{\left(k_1{}^2 + k_2{}^2\right)}_z \qquad ...(5.4.4)$$

Putting the values of (5.4.4) in equation (5.4.2) we get,

$$\phi = XYZ$$

$$= \left(Ae^{k_1x} + Be^{-k_1x}\right)\left(Ce^{k_2y} + De^{-k_2y}\right)$$

$$\left(E\cos\sqrt{\left(k_1{}^2 + k_2{}^2\right)}z + F\sin\sqrt{\left(k_1{}^2 + k_2{}^2\right)}z\right) \qquad ...(5.4.5)$$

Now, if we take,

$$\frac{1}{X}\frac{d^2X}{dx^2} = k_1{}^2$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = k_2{}^2$$

and $$\frac{1}{z}\frac{d^2Z}{dz^2} = -\left(k_1^2 + k_2^2\right) = -k_3^2$$

Then solution of equation (5.4.2) becomes

$$\phi = X\ Y\ Z$$

$$= (A \cos k_1x + B \sin k_1x)\ (C \cos k_2y + D \sin k_2y)$$

$$\left(Ee^{-\sqrt{k_1{}^2+k_2{}^2}z} + Fe^{-\sqrt{k_1{}^2+k_2{}^2}z}\right)$$

II. In cylindrical co-ordinate system (r, θ, z)

$$x = r\cos\theta$$

$$y = r\sin\theta$$

and $$z = z$$

Therefore, Laplace's equation in cylindrical form is

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial z^2}=0 \qquad ...(5.4.6)$$

Let the solution is

$$\phi\ (r,\ \theta,\ z) = R\ (r)\ \ Q\ (\theta)\ Z(z) \qquad ...(5.4.7)$$

Substituting equation (5.4.7) in (5.4.6) we get,

$$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)+\frac{1}{r^2Q}\frac{d^2Q}{d\theta^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=0 \qquad ...(5.4.8)$$

Let, $\frac{1}{Z}\frac{d^2Z}{dz^2}=k^2$

and $\frac{1}{Q}\frac{d^2Q}{d\theta^2}=-n^2$

$$\left.\begin{aligned}
&\text{and} && \frac{1}{Q}\frac{d^2Q}{d\theta^2}=-n^2\\
&\text{Then,} && \frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)-\frac{n^2}{r^2}+k^2\Bigg)=0\\
&\text{or,} && r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+\left(k^2r^2-n^2\right)R=0
\end{aligned}\right\} \qquad ...(5.4.9)$$

From, equation (5.4.9), we obtain the solution

$Z = (E\ e^{kz} + F\ e^{-kz})$

$Q = (C \cos n\theta + D \sin n\theta)$

and $R = A\ J_n\ (kr) + B\ J_{-n}\ (kr)$, where

$J_n(kr)$ and $J_{-n}(kr)$ are Bessel's function and complete solution of

$$\frac{1}{r^2}\frac{d^2R}{dr^2}+\frac{rd^2R}{dr}+\left(k^2r^2-n^2\right)R=0$$

Should be same as equation (2.6.1)

Hence, equation (5.3.7) becomes

$\phi\ (r, \theta, z) = [A\ J_n\ (k_r) + B\ J_{-n}\ (k_r)]\ (C \cos n\theta + D \sin n\theta)\ (E\ e^{kz} + F\ e^{-kz})$ ..... (5.4.10)

This solution is known as the cylindrical harmonics.

III. In spherical polar coordinate system (r, θ, ϕ)

$x = r \sin \theta \cos \phi$

$y = r \sin \theta \sin \phi$

$z = r \cos \theta$

So, Laplace's equation in spherical polar coordinate system becomes

$$\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\phi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} = 0 \qquad ...(5.4.11)$$

Let, the solution be

$$\phi\ (r, \theta, Q) = R\ (r)\ Q(\theta)\ F(\varphi) \qquad ...(5.4.12)$$

Substituting equation (5.4.12) in (5.4.11) we get,

$$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr}\right) + \frac{1}{Q}\left(\frac{d^2Q}{d\theta^2} + \cot\theta\frac{dQ}{d\theta}\right) + \frac{1}{F\sin^2\theta}\frac{d^2F}{d\varphi^2} = 0 \qquad ...(5.4.13)$$

Let us consider, $\frac{1}{R}\left(r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr}\right) = n(n+1))$ ...(5.4.14)

and $\frac{1}{F}\frac{d^2F}{d\varphi^2} = -m^2$ ...(5.4.15)

Where $-m^2$ and $n\ (n + 1)$ are the two seperation constants.

Hence, $\dfrac{d^2Q}{d\theta^2}+\cos\theta\dfrac{dQ}{d\theta}+\left\{n(n+1)-\dfrac{m^2}{\sin^2\theta}\right\}Q=0$

This equation is called the associated Legender's equation. ... (5.4.16)

From equation (5.4.15), we get,

$F = C \cos m\varphi + D \sin m\varphi$ ...(5.4.17)

and from equation (5.4.14) we get,

$k\ (k - 1) + 2k = n\ (n - 1)$ [By putting $R = r^k$]

or, $k = n$ or, $k = -\ (n + l)$

Hence, $R = E\ r^n + F\ r^{-(n+1)}$ ...(5.4.18)

The solution of

equation (5.4.16) can be written as

$Q = A\ P_n^m \cos\theta + B\ Q_n^m \sin\theta$

Hence, equation (5.4.12) becomes

$$\phi=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(Er^n+\frac{F}{r^{n+1}}\right)(C\cos mq+D\sin mq)$$

$$(A\ P_n^m \cos\theta + B\ Q_n^m \sin\theta) \qquad ...(5.4.19)$$

This solution is known as spherical harmonics.

5.4.1 Illustrated examples :

(1) Find the solution of the given equation

$\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}=0$; inside the annular regions bounded by the circles $x^2 + y^2 = r_1^2$ and $x^2 + y^2 = r_2^2$ that satisfies the conditions

$\phi = \phi_1$ at $r = r_1$ and

$\phi = \phi_2$ at $r = r_2$

Solution. The said equation is

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

This is two-dimentional Laplace's equation in certision form.

For two-dimensional polar coordinate, the equation can be written as

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}=0$$

or, $$\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)=0$$

or, $$r\frac{\partial\phi}{\partial r}=A,\ \text{constant}$$

or, $$\partial\phi = A\frac{\partial r}{r}$$

Integrating bothsides we get $\phi = A \ln r + B$ (constant). Now put the boundary value conditions in the above equationn and we get,

$$\phi_1 = A \ln r_1 + B$$

$$\phi_2 = A \ln r_2 + B$$

Equating the equations we obtain,

$$A\ln\left(\frac{r_1}{r_2}\right)=\phi_1-\phi_2$$

$$\therefore\ A=\frac{\phi_1-\phi_2}{\ln\left(\frac{r_1}{r_2}\right)}$$

$$\text{and B} = \phi_1 - \frac{\phi_1 \ln r_1 - \phi_2 \ln r_1}{\ln r_1 - \ln r_2}$$

$$= \frac{\phi_2 \ln r_1 - \phi_1 \ln r_2}{\ln\left(\frac{r_1}{r_2}\right)}$$

Hence we get, the solution of the given equation is

$$\phi = \frac{\phi_1 - \phi_2}{\ln\left(\frac{r_1}{r_2}\right)} \ln r + \frac{\phi_2 \ln r_1 - \phi_1 \ln r_2}{\ln\left(\frac{r_1}{r_2}\right)}$$

2. Suppose the following equation refers to a problem of two-dimensional steady flow of heat :

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

Boundary conditions are

$$\psi\ (0,\ y) = 0,\ \psi\ (x,\ \infty) = 0$$

$$\psi\ (a,\ y) = 0,\ \psi\ (x,\ 0) = \sin\frac{\pi x}{a}$$

Solution. Using the method of separation of variables, the solution can be written as

$$\psi = \psi_1\ (x)\ \psi_2\ (y) \qquad ...(1)$$

Putting the equation (1) in said equation we obtain,

$$\text{or, } \psi_2\, \psi_1'' + \psi_1\, \psi_2'' = 0$$

$$\text{or, } \frac{\psi_1''}{\psi_1'} = -\frac{\psi_2''}{\psi_2} = -k^2\, (\text{say})$$

$$\therefore \quad \psi_1'' = -k^2 \psi_1 \qquad ...(2)$$

$$\text{and } \psi_2'' = k^2 \psi_2 \qquad ...(3)$$

From equation we get, $\psi_1 = A \sin kx + B \cos kx$

and from equation (3) we get, $\psi_2 = C\, e^{ky} + D\, e^{-ky}$

Hence, equation (1) becomes

$$\psi(x, y) = (A \sin k x + B \cos k x)(C e^{ky} + D e^{-ky})$$
$$= e^{ky}(A_1 \sin k x + B_1 \cos k x)$$
$$+ e^{-ky}(A_2 \sin k x + B_2 \cos k x) \quad ...(4)$$

Using boundary condition $\psi(0, y) = 0$, in equation (4); we get,

$B_1 = B_2 = 0$ and equation (4) becomes

$$\psi(x, y) = A_1 e^{ky} \sin kx + A_2 e^{-ky} \sin kx \quad ...(5)$$

Using boundary condition $\psi(x, \infty) = 0$, in equation (5) we get,

$A_1 = 0$ and equation (5) becomes

$$\psi(x, y) = A_2 e^{-ky} \sin kx \quad ...(6)$$

Using boundary condition $\psi(a, y) = 0$, in equation (6) we get

$$\sin a k = 0 \Rightarrow ka = n\pi, \; [n = 0, 1, 2, 3...]$$

and equation (6) becomes $\psi(x, y) = A_2 e^{-ky} \sin \frac{n\pi}{a} x \quad ...(7)$

Using boundary condition $\psi(x, 0) = \sin \frac{\pi x}{a}$ in equation we get,

$\sin \frac{n\pi x}{a} = \sin \frac{\pi x}{a} \Rightarrow A_2 = 1$ and $n = 1$, Hence the required solution is

$$\boxed{\psi(x, y) = e^{-ky} \sin \frac{\pi x}{a}} \; ......$$

5.5 Wave equation :

The general form of wave equation is

$$\nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} \quad ...(5.5.1)$$

For one-dimensional case equation (5.5.1) becomes

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} \qquad ...(5.5.2)$$

Here [φ (*x*, *t*)] has two independent variables *x* and *t*. Hence seperation of variables method gives the solution.

$$\phi = X(x)\, T(t) \qquad ...(5.5.3)$$

Putting equation (5.4.3) in equation (5.4.2) we obtain

$$T\frac{d^2X}{dx^2} = \frac{1}{c^2} \times \frac{d^2T}{dt^2}$$

$$\text{or,} \quad \frac{\frac{d^2X}{dx^2}}{X} = \frac{\frac{d^2T}{dt^2}}{Tc^2} = -k^2 \text{ (say)}$$

$$\text{So,} \quad \frac{d^2X}{dx^2} + k^2X = 0$$

$$\text{and} \quad \frac{d^2T}{dt^2} + c^2k^2T = 0 \qquad ...(5.5.4)$$

The solution. of equation (5.4.4) are

$$X = A \cos kx + B \sin kx$$

$$\text{and } T = C \cos ckt + D \sin ckt$$

Hence equation (5.4.3) becomes

$$\phi = (A \cos kx + B \sin kx)(C \cos ckt + D \sin ckt) \qquad ...(5.5.5)$$

For 3-dimentional case,

$\phi = X(x)\, Y(y)\, Z(z)\, T(t)$, so equation (5.4.5) is giving by the product of *X*, *Y*, *Z*, *T* and

$\phi = (A_1 \cos k_1x + B_1 \sin k_1x)\,(A_2 \cos k_2y + B_2 \sin k_2y)\,(A_3 \cos k_3z + B_3 \sin k_3z)\,(C \cos ckt + D \sin ckt)$ ...(5.5.6)

The equation (5.5.6) can be written in a different form

$$\phi = Ae^{\pm i\left(ckt \pm k_1x \pm k_2y \pm k_3z\right)} \quad \text{...(5.5.7)}$$

For real k_1, k_2, k_3, the above solution represents a plane prograssive wave in the direction of the wave vector $\vec{k} = k_1\hat{i} + k_2\hat{j} + k_3\hat{k}$ and c the velocity with which the wave travels.

5.5.1 Solution for vibrational modes of a stretched string :

For stretched string, two nodes should be produced at the end point, that means if the string has length l, then we can apply the following boundary conditions:

(i) $\phi = 0$ at $x = 0$

(ii) $\phi = 0$ at $x = l$

So for one-dimensional case, equation (5.4.5) gives

$$A\,(C \cos ckt + D \sin ckt) = 0 \Rightarrow A = 0$$

and $(C \cos ckt + D \sin ckt)\,(A \cos kl + B \sin kl) = 0$

$\Rightarrow \sin kl = 0$ or, $kl = n\pi$ (n is integer) $[\because B \neq 0]$

Hence equation (5.5.5) becomes

$$\phi = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left[C_n \cos\frac{n\pi ct}{l} + D_n \sin\frac{n\pi ct}{l}\right] \quad \text{...(5.4.8)}$$

Let us apply the initial cndition i.e.,

(i) at $t = 0$,

$$\phi = f(x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} \quad \text{...(5.4.9)}$$

(ii) at $t = 0$, $\frac{\partial\phi}{\partial t} = 0$

So, $$\sum_{n=1}^{\infty} \frac{n\pi c}{l} \sin\frac{n\pi x}{l} D_n = 0$$

$\Rightarrow$ $D_n = 0$ for all values of n.

Hence, equation (5.5.8) becomes

$$\phi = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad ...(5.5.10)$$

Where, $$C_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

[From Fourier expansion of the function $f(x)$]

5.5.2 Two dimensional equation

From equation (5.5.1) we obtain the wave equation

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{dt^2}$$

For two-dimension case, Let us assume the solution.

$$\phi(x, y, t) = X(x)\,Y(y)\,T(t) \qquad ...(5.5.11)$$

Putting the value of equation (5.4.11) in equation (5.5.11) we get,

$$\frac{1}{c^2}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} \qquad ...(5.5.12)$$

Since x, y and t are independent. Hence we can write the equation (5.5.12) by choosing proper constants,

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k_x^2$$

or, $\dfrac{d^2X}{dx^2} + k_x X = 0$ and $\dfrac{1}{Y}\dfrac{d^2Y}{dy^2} = -ky^2$ ...(5.5.13)

or, $\dfrac{d^2Y}{dy^2} + k_y^2 Y = 0$

Now, say $k_x^2 + k_y^2 = k^2$, then

$$\frac{1}{C^2}\frac{d^2T}{dt^2} = -k^2$$

or, $\dfrac{d^2T}{dt^2} + k^2c^2T = 0$ ...(5.5.14)

From equation (5.5.13) and (5.5.14) we get,

$\phi\ (x, y, t) = (A \cos k_x x + B \sin k_x x)\ (C \cos k_y y + D \sin k_y y)$

$(E \cos ckt + F \sin ckt)$ ...(5.5.15)

For rectangular membrane, boundary conditions are

(i) $\phi = 0$ when $x = 0$ (ii) $\phi = 0$ when $x = a$

(ii) $\phi = 0$ when $y = 0$, (iv) $\phi = 0$ when $y = b$

Substituting condition (i) in equation (5.5.15) we get,

$$A\ (C \cos k_y y + D \sin k_y y)\ (E \cos ckt + F \sin ckt) = 0$$

$$\Rightarrow A = 0$$

Now, equation (5.5.15) becomes

$\phi\ (x, y, t) = B \sin k_x x\ (C \cos k_y y + D \sin k_y y)$

$(E \cos ckt + F \sin ckt)$ ...(5.5.16)

Substituting condition (ii) in equation (5.5.16) we get,

$\sin k_x a = 0$, As $B \neq 0$

$\Rightarrow\ k_x = \dfrac{n\pi}{a}$, $n = 1, 2, 3$ .....

Now, equation (5.5.16) becomes

$$\phi\ (x, y, t) = B \sin \frac{n\pi}{a} x\ (C \cos k_y y + D \sin k_y y)$$

$$(E \cos ckt + F \sin ckt) \qquad ...(5.5.17)$$

Substituting the boundary condition (iii) and (iv) in equation (5.5.17), we obtain

$C = 0$ and $ky = \frac{m\pi}{b}$, m = 1, 2, 3 .....

Hence,

$$\phi\ (x, y, t) = BD \sin \frac{n\pi}{a} x \sin \frac{m\pi}{b} y \quad (E \cos ckt + F \sin ckt)$$

$$= BD \sin \frac{n\pi}{a} x \sin \frac{m\pi}{b} y \quad (E \cos \omega t + F \sin \omega t) \qquad ...(5.5.18)$$

Where, $\omega = ck = c\sqrt{k_x^2 + k_y^2}$

$$= \pi c \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$$

Replacing the arbitrary constants in equation (5.5.18)

we get $\phi\ (x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sin \frac{n\pi}{a} x \sin \frac{m\pi}{b} y \ (E_{mn} \cos \omega t + F_{mn} \sin \omega t)$ (5.5.19)

Again, from the initial condition $\phi = f(x, y)$ at $t = 0$

and $\frac{\partial \phi}{\partial t} = 0$ at $t = 0$, we obtain,

$F_{mn} = 0$ and

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} E_{mn} \sin\frac{n\pi}{a} x \sin\frac{m\pi}{b} y \qquad ...(5.5.20)$$

The above expression is a double Fourier series.

Where,

$$E_{mn} = \left[\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi}{a} x\,dx\right]$$

$$\left[\frac{2}{b}\int_0^b f(y)\sin\frac{m\pi}{b} y\,dy\right]$$

$$= \frac{4}{ab}\int_0^a\int_0^b f(x)f(y)\sin\frac{n\pi}{a} x \sin\frac{m\pi}{b} y\,dx\,dy$$

$$= \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\frac{n\pi}{a} x \sin\frac{m\pi}{b}\,\mathrm{y}$$

[From double Fourier sine series]

Another boundary condition can be taken into account, when the rectangular membrance defined within $\{0 < x < a, 0 < y < b\}$ and has initial displacement

$f' \to f(x, y)$ and initial velocity $f'(x, y)$.

In this case, the velocity function

$$f'(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\,\omega \sin\frac{n\pi}{a} x \sin\frac{m\pi}{a} y \qquad ...(5.5.21)$$

Where,

$$B_{mn} = \frac{4}{ab\omega}\int_0^b\int_0^a f'(x,y)\sin\frac{n\pi}{a} x \sin\frac{m\pi}{b} y\,dx\,dy$$

5.6 Summary :

Method of separation of variables to solve partial differential equations such as Laplace's equation and Wave equation has been discussed.

5.7 Review Questions and Answer :

(1) Solve the boundary value problem

$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$, given $u\,(0,\, y) = 8e^{-3y}$ by the method of seperation of variables.

Ans. Let the solution of the given equation is

$$u = \psi\,(x)\,\phi\,(y) \qquad ...(1)$$

Putting equation (1) in the given equation we obtain,

$$\phi\frac{\partial\psi}{\partial x} = 4\psi\frac{\partial\phi}{\partial y}$$

$$\text{or, } \frac{1}{4\psi}\frac{d\psi}{dx} = \frac{1}{\phi}\frac{d\phi}{dy} = k \text{ (say)}$$

$$\therefore \frac{d\psi}{\psi} = 4k\,dx \text{ and } \frac{d\phi}{\phi} = k\,dy$$

Integrating both equations we get,

$ln\ \psi = 4kx + lnA$ and $ln\,\phi = k\,y + lnB$

$$\therefore \psi = Ae^{-4kx} \qquad ...(2)$$

$$\text{and } \phi = Be^{-k}y \qquad ...(3)$$

Putting (2) and (3) in equation (1) we get,

$$u\,(x,\,y) = C\,e^{-4kx}\,e^{-ky} \qquad ...(4)$$

Using boundary condition in equation (4) we get,

$$C\,e^{-ky} = 8\,e^{-3y}$$

$\Rightarrow C = 8$ and $k = 3$ and equation (4)

becomes

$$u(x,\,y) = 8\,e^{-12x}\,e^{-3y}$$

or, $\boxed{u(x, y) = 8e^{-(12x+3y)}}$

This is the required solution.

2. Write down the three-dimentional equation of heat flow through a medium of thermal conductivity *K*. Show that under steady conditions this equation reduces to Laplace's equation $\nabla^2 \phi = 0$, *w* were ϕ (*r*) denotes the temperature and can be written as

$$\phi(r) = \frac{Q}{4\pi kr} + B$$

Ans. Three-dimensional equation of heat flow through a medium is, given by

$\frac{\partial \phi}{\partial t} = \alpha^2 \nabla^2 \phi$, where $\alpha = \frac{K}{\rho s}$ is the diffussivity of the medium of density ρ, Specific leat s.

Under steady condition, $\frac{\partial \phi}{\partial t} = 0 \Rightarrow$

$\nabla^2 \phi = 0$, well known by Laplace's equation.

In spherical polar co-ordinates (r, θ, ϕ), the above equation can be written as

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial \phi}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \phi}{\partial \varphi^2} = 0 \qquad \text{[From equation (5.4.11)]}$$

Now, heat only flows radially in outward direction. Hence the above equation must be independent of θ and φ.

$$\therefore \quad \frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r} = 0$$

$$\text{or,} \quad \frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) = 0$$

or, $r^2 \frac{d\phi}{dr} = A$ (constant)

or, $d\phi = A \frac{dr}{r^2}$

Integrating bothsides we get,

$$\phi = -\frac{A}{r} + B \text{ (constant)}$$

From the heat conduction eqnation we get $Q = -K\, 4\pi r^2 \frac{d\phi}{dr}$

or, $Q = -K\, 4\pi r^2 \left(\frac{A}{r^2}\right)$

$\therefore \quad A = -\frac{Q}{4\pi K}$

Thus we obtain,

$$\phi(r) = \frac{Q}{4\pi kr} + B$$

3. A rectangular stretched membrane of sides *a* and *b* having edges parallel to the *x-axis* and *y-axis*, and bounded rigidly at the edges, is given a slight deformation in 2-direction perpendicular to its own plane. The differential equation for *z* is

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}, \text{ where } c \text{ is constant.}$$

Solve the equation by the method of separation of variables assuming the initial condition

$$\phi = f(x, y) \text{ and } \frac{\partial\phi}{\partial t} = 0 \text{ at } t = 0.$$

Ans. See the solution for the two-dimentional equation discussed in the article no. (5.5.2).

Unit – 6 ❑ Advance Mechanics

6.0 Objectives

1. Classification of constraints with some examples, degrees of freedom, generalized coordinates, generalised displacement, virtual work.
2. D'Alembert's participle, Euler-Lagrange's equations with application.
3. Concept of symmetry, conservation of linear and angular momentum and conservation of energy.
4. Hamiltonian formulation of mechanics and its advantage and applications.

6.1 Introduction :

Mechanics deals with the motion of physical objects, whether large or small. Advance Mechanics is the alternative way, by which we can express the Newton's laws of motion. Advance Mechanics donate that part of mechanics where the objects are neither too big nor too small interacting objects. Classical (advance) mechanics address a huge range of problems ranging from molecular dynamics to the motion of celestial bodies. Classical mechanics is very useful for analysing problems in which quantum and relativistic effects are negligible. Classical mechanics has many applications in the areas of Astronomy,dynamics of molecular collissions, propagation of seismic waves, generated by earthquakes etc.

6.2 Constraints :

Constant is defined as a restriction to the fredom in the motion of an object. A motion under constraints is known as constrained motion. Motion along a specified path is the simplest example of a constrained motion. Imposing constraints on a mechanical system is done to simplify the mathematical description of the system. The number of coordinates needed to specify the dynamical system becomes smaller when constraints are present in the system. Hence the degree of freedom of a dynamical system is defined as the minimum number of independent coordinates required to simplify the system completely along with the constraints. Thus if k is the number of constraints and N is the number of particles in the system possessing motion in 3-dimensions then the number of degree of freedom are give by

$n = 3N - k$ ... (6.1), where n is the number of degrees of freedom.

6.2.1 Classification of Constraints :

There are many types of constraintts. A constraints is said to be

1. **Scleronomic :** If the constraint relations does not depend on time explicitly.
2. **Rheonomic :** If the constraint relation depends on time explicitly.
3. **Holonomic :** Constraints expressed in the form of equation $f(r_1, r_2 \ldots\ldots r_n ; t) = 0$ are called holonomic ... (6.2)
4. **Non-holonomic :** Constraints that can not be expressed in the form of equation (6.2) are called non-holonomic.
5. **Conservative :** If the total energy of the system is conserved during the constrained motion, the constraints are called conservative.
6. **Dissipative :** If the total energy of the system is not conserved, the constraints are dissipative.
7. **Bilateral and Unilateral :** If constraint relations are in the form of equations, they are bilateral, but if relations are expressed in the form inequalites then constraints are unilateral.

6.2.2 Example of constraints :

1. **Rigid body :** In a rigid body the distances any pairs of particles are constant i.e.

$$\left|\vec{r}_i - \vec{r}_j\right| = \text{constant}$$

$\vec{r}_i$ and $\dot{\vec{r}}_j$ are the position vectors of i-th and j-th particles of the object.

The constraint is conservative, scleronomic, holonomic, and bilateral.

2. **Deformable bodies :**

For a deformable bodies, the distance between any pair of particular change in time, so

$$\left|\vec{r}_1 - \vec{r}_j\right| = f(t) \text{ for all } i \text{ and } j.$$

The constraint is holonomic, bilateral, dissipative and Rheonomic.

3. Simple pendulum with rigid support :

The position of the bob satisfy the relation

$\left|\vec{r}\right| = l$ where, l is the constant length of the pendulum.

The constraint is scleronomic, holonomic, bilateral and conservative.

4. Pendulum with variable Length :

Equation of constraint is $|\vec{r}(t)|^2 = l^2(t)$ Hence, constraint is Rheonomic, holonomic, bilateral and dissipative.

5. Rolling without sliding : Frictional force without sliding do not work and total mechanical energy is conserved. Hence constraint is conservative and non-holonomic.

6. Gas filled hollow sphere : Gas molecules are constrained by the walls and can only move inside the sphere of radius R(say). So the constrains relation are

$\left|\vec{r_i}\right| \le R$.

The constraint is scleronomic, holonomic, unilateral and conservative.

7. Expanding or contracting gas filled container :

The dimension of container is changing with time, so, $\left|\vec{r_i}\right| \le R(t)$

The constraint is rheonomic, holonomic, unilateral and dissipative.

6.3 Degrees of freedom :

Degrees of freedom is the minimum number of independent variables (q_1, q_2, ... q_n, say) that is necessary to fix uniquely the position and the configuration of the given system, compatible with the given constraints.

For a free particle, in three dimension space, the degrees of freedom can be specified by its three position coordinates, so number of degrees of freeding is 3.

Henec, for N number of particles, degrees of freedom is 3N. Now if K number of constraint relations are their in the system, the degrees of freedom reduces to 3N – K.

Example 1. Triatomic molecule linearly arranged :
Here $N = 3$ and no. of constraints $K = 2$ [fixed distances between]. So, degrees of freedom is 3.3 – 2 = 7.

Example 2. The bob of conical pendulum :
Here the constraint is the fixed Legth l. Hence number of degrees of freedom is 3.1 – 1 = 2.

Example 3. A dumbbell : A dumbbell consists of two heavy point particles connected to each other by a mass less rigid rod of length l. So, degrees x of freedom is 3.2–1 = 5

6.4 Generalised coordinates :

To describe the configuration of a system, we consider the smallest possible number of variables, which are called the generalised coordinates of the system. A set of generalized co-ordinates is any set of co-ordinates which describe the configuration.

How to choose a suitable set of generalised co-ordinates in a given situation? By three principles, we can solve the problem :

(i) Their values determine the configuration of the system.

(ii) They may be varied arbitrarily and independently of each other, without violating the constraints on the system.

(iii) There is no uniqueness in the choice of generalized coordinates. Then our choice should fall on a set of co-ordinates that will give us a reasonable mathematical simplification of the problem.

For a system of N particles with K independent constaints the number of independent variables to specify the configuration is $n = 3N - K$.

Notation for generalised co-ordinates : Generalised co-ordinates are denoted by q with numerical subscripts : $q_1, q_2, ... q_n$ represent a set of n generalised co-ordinates.

In general we can express generalised co-ordinates as function of certain co-ordinates and time i.e.,

$q_1 = q_1(x_1, y_1, z_1 \; ; \; .... \; ; \; t)$

$q_2 = q_2 \; (x_1, y_1, z_1, x_2, y_2, z_2, \; ; \; .... \; , \; t)$

$q_n = q_n \; (x_1, y_1, z_1 \; ; \; ....x_n, y_n, z_n, t)$

When a particle moves in a plane, it may be described by cartesian co-ordinates x, y and we write

$$\begin{cases} q_1 = x \\ q_2 = y \end{cases}$$

For spherical symmetry, $\begin{cases} q_1 = r \\ q_2 = \theta \\ q_3 = \phi \end{cases}$

6.4.1 Generalised Displacement :

Let us consider a system of N particular have n degrees of freedom. The position vector in terms of generalised co-ordinate is defined

as $\vec{r_i} = \vec{r_i}\left(q_1, q_2, ...q_n, t\right)$

... (6.4.1)

Hence, the small displacement, where $n = 3N - K$

$$\partial\vec{r_i} = \sum_{j=1}^{n} \frac{\partial \vec{r_i}}{\partial r_j} \partial q_j + \frac{\partial \vec{r_i}}{\partial t} \partial t$$

as $\partial t = 0$, so $\partial \vec{r_i} = \sum_{j=1}^{n} \frac{\partial \vec{r_i}}{\partial q_j} \partial q_j$ ... (6.4.2)

∂q_j are called the generalised displacements or virtual displacement. This change in the system is not associated with a change in time *i.e*, there is no actual displacement and hence the displacement is termed as virtual displacement.

6.4.2 Virtual Work :

We first have to define the generalised force, associated with generalised displacement.

Let us consider the total force $\sum F_i$ acts on a system during a small displacement $\partial \vec{r_i}$.Then total work done.

$$\partial W = \sum_{i=1}^{N} \vec{F_i} \cdot \partial \vec{r_i}$$

$$= \sum_{i=1}^{N} F_i \cdot \sum_{j=1}^{n} \frac{\partial \vec{r_i}}{\partial q_j} \partial q_j$$

$$= \sum_{j=1}^{n} Q_j \partial q_j \quad \text{... (6.4.3)}$$

where $Q_j = \sum_{i=1}^{N} \vec{F_i} \cdot \frac{\partial \vec{r}}{\partial q_j}$ ... (6.4.4) is

the generalised force associated with co-ordinate q_j. The product of Q_j ; with the generalised displacement dq_j is equal to the work done corresponding to the displacement. From equation (6.4.3), it follows that the product of dimension of generalised displacement and dimension of generalised force must have the dimension of work. As the displacement is virtual so, the work done for virtual displacement is called virtual work.

6.5 D'Alembert's principle :

This method is based on the principle of virtual work. Let us consider a system with equilibrium condition i.e., total force $\sum F_i$ on each particles are zero, then work done by the force in a small displacement $\delta \vec{r_i}$ will also zero. Hence for whole system of N particles

$$\sum_{i=1}^{N} F_i \cdot \delta\vec{r_i} = 0 \qquad \text{... (6.5.1)}$$

Let the total force $\vec{F_i}$ is the sum of applied force $\vec{F}_i^{(a)}$ and forces of constraints $\vec{F}_i^{(c)}$, then equation (6.5.1) becomes

$$\sum_{i=1}^{N} \vec{F}_i^{(a)} \cdot \delta\vec{r_i} + \sum_{i=1}^{N} \vec{F}_i^{(c)} \cdot \delta\vec{r_i} = 0$$

Virtual work done by the constraints is zero.

so $\sum_{i=1}^{N} \vec{F}_i^{(a)} \cdot \delta\vec{r_i} = 0$ .... (6.5.2)

The above equation is called principle of virtual work. To specify the equilibrium system. D' Alembert taken an idea of reversed force. When the applied force $\vec{F}_i^{(a)}$ and the reversed force $\dot{\vec{P}}_i$ are equal then the system showd be in equilibrium condition i.e.,

$$\vec{F}_i^{(a)} = \dot{\vec{P}}_i$$

or, $\vec{F}_i^{(a)} - \dot{\vec{P}}_i = 0$ and equation (6.5.2) becomes

$$\sum_{i=1}^{N} \left(\vec{F}_i^{(a)} - \dot{\vec{P}}_i \right) \cdot \delta\vec{r_i} = 0 \qquad \text{... (6.5.3)}$$

The equation (6.5.3) is called D'Alembert's principle. D'Alemmbert's principle is just one equation of motion. Since the constraint forces do not appear, it is sufficient if only applied forces are specified and also reverse effective force $\left(\vec{P_i} \right)$ reduces the problems of dynamics to that of statics.

6.5.1. Derivation of Euler-Lagrange's equations :

From equation (6.4.1) we get,

$$\vec{r_i} = \vec{r_i}\left(q_1, q_2 \ldots\ldots\ldots q_n, t\right)$$

The velocity of the i-th particle is given by

$$\vec{v_i} = \frac{d\vec{r_i}}{dt} = \sum_j \frac{\partial \vec{r_i}}{\partial q_j}\frac{\partial q_j}{\partial t} + \frac{\partial \vec{r_i}}{\partial t} \qquad \ldots(6.5.4)$$

Further infinitesimal diplacement $\partial\vec{r_i}$ can be written as

$$\delta\vec{r_i} = \sum_{i=1}^{n} \frac{\partial \vec{r_i}}{\partial q_j}\delta q_j + \frac{\partial \vec{r_i}}{\partial t}\delta t$$

For virtual displacement, $\delta t = 0$, hence

$$\delta\vec{r_i} = \sum_{j=1}^{n} \frac{\partial \vec{r_i}}{\partial q_j}\delta q_j \quad \ldots (6.5.5)$$

Putting equation (6.5.5) in equation (6.5.3) we obtain,

$$\sum_{i=1}^{N}\left(\vec{F}_i^{(a)} - \dot{\vec{P}}_i\right)\cdot\sum_{j=1}^{n} \frac{\partial \vec{r_i}}{\partial q_j}\delta q_j = 0$$

or, $$\sum_{i=1}^{N}\sum_{j=1}^{n}\left(\vec{F_i} - \dot{\vec{P}}_i\right)\cdot\frac{\partial \vec{r_i}}{\partial q_j}\delta q_j = 0$$ [Just dropping the superscript a]

$$\text{or,} \quad \sum_{i_j} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j - \sum_{i_j} \vec{P}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j = 0$$

$$\text{or,} \quad \sum_j Q_j \partial q_j \underset{j}{} - \sum_{i_j} \vec{P}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j = 0$$ [From equation (6.4.4)]

.... (6.5.6)

The 2nd term can be written as

$$\sum_{i_j} \left(\dot{\vec{P}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) \delta q_j = \sum_{i_j} m_i \cdot \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$

$$= \sum_{i_j} \frac{d}{dt} \left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_j}{\partial q_j} \right) \delta q_j - \sum_{i_j} m_i \cdot \dot{\vec{r}} \cdot \frac{d}{dt} \left(\frac{\partial \vec{r}_i}{\partial q_j} \right) \delta q_j \qquad \text{... (6.5.7)}$$

$$\text{Now,} \quad \frac{d}{dt} \left(\frac{\partial \vec{r}_i}{\partial q_j} \right) = \sum_K \frac{\delta}{\delta q_K} \left(\frac{\partial \vec{r}_i}{\partial q_j} \right) \cdot \frac{dq_K}{dt} + \frac{d}{dt} \left(\frac{\partial \vec{r}_i}{\partial q_j} \right)$$

$$= \sum_K \frac{\partial^2 \vec{r}_i}{\partial q_k \partial q_j} \frac{\partial q_k}{\partial t} + \frac{d}{dt} \left(\frac{\partial \vec{r}_i}{\delta q_j} \right)$$

$$= \frac{\partial}{\partial q_j} \left[\sum_K \frac{\partial \vec{r}_i}{\partial q_k} \frac{\partial q_k}{\partial t} + \frac{\partial \vec{r}_i}{\partial t} \right]$$

$$= \frac{\partial \dot{\vec{r_i}}}{\partial q_j} \text{ [from equation (6.5.4)]}$$

or, $$\frac{d}{dt}\left(\frac{\partial \vec{r_i}}{\partial q_j}\right) = \frac{\partial \vec{v_i}}{\partial q_j}$$

From equation (6.5.4) we get,

$$\vec{v_i} = \sum_j \frac{\partial \vec{v_i}}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r_i}}{\partial t}$$

and, $$\frac{\partial \vec{v_i}}{\partial \dot{q}_j} = \frac{\partial \vec{r_i}}{\partial q_j}$$

Hence, equation (6.5.7) becomes

$$\sum_{i_j} \dot{\vec{P}}_i \cdot \frac{\partial \vec{v_i}}{\partial \dot{q}_j}\delta q_j = \sum_{i_j}\left\{\frac{d}{dt}\left(m\vec{v_i}\cdot\frac{\partial \vec{v_i}}{\partial \dot{q}_j}\right) - m_i \cdot \vec{v_i}\cdot\frac{\partial \vec{v_i}}{\partial q_j}\right\}\delta q_j$$

$$= \sum_{i_j}\left[\frac{d}{dt}\left\{\frac{d}{d\dot{q}_j}\left(\sum_i \frac{1}{2}m_i v_i^2\right)\right\} - \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2}m_i v_i^2\right)\right]\delta q_j$$

$$= \sum_{i_j}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right]\delta q_j$$

where $T = \sum \frac{1}{2} m_i \cdot v_i^2$ represents the kinetic energy of i the system.

Now, from equation (6.5.6) we get,

$$\sum_i Q.\delta q_j - \sum_j\left[\frac{d}{dt}\left(\frac{\partial T}{\partial q_j}\right) - \frac{\partial T}{\partial q_j}\right]\delta q_j = 0$$

$$\text{or, } \sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial q_j}\right) - \frac{\partial T}{\partial q_j} - q_j\right]\delta Q_j = 0$$

Since, q_j are independent to each other hence the co-efficient of δq_j should be zero i.e.,

$$\sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial q_j}\right) - \frac{\partial T}{\partial q_j} - Q_j\right] = 0$$

$$\text{or, } \sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial q_j}\right) - \frac{\partial T}{\partial q_j}\right] = \sum Q_j \qquad \text{... (6.5.8)}$$

Case I. Conservative system :

For conservative system, forces F_i are derivable from potential function $V = V(q_j)$ i.e., only dependent on co-ordinates.

So, we can write, $\vec{F_i} = -\vec{\nabla}_i V = -\frac{\partial V}{dr_j}$

$$\text{or, } Q_j = \sum_i \vec{F_i}\cdot\frac{\partial \vec{r_i}}{\delta q_j} = -\sum_i \vec{\nabla}_i V \cdot \frac{\partial \vec{r_i}}{\partial q_j}$$

$$= -\sum_i \frac{\partial V}{\partial \vec{r_i}} \cdot \frac{\partial \vec{r_i}}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

Equation (6.5.8) becomes

$$\sum_i \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q_j}\right] = \sum_j -\frac{\partial V}{\partial q_j}$$

or, $\sum_i \left[\frac{d}{dt} \cdot \frac{\partial (T-V)}{\partial \dot{q}_j} - \frac{\partial (T-V)}{\partial \dot{q}_j} \right] = 0$ [since V is not the function of $\dot{q}_j$]

or, $\sum_j \left[\frac{d}{dt} \left(\frac{\partial f}{\partial \dot{q}_j} \right) - \frac{\partial f}{\partial q_j} \right] = 0$ ... (6.5.9)

(6.5.9) is known as Euler Lagrange equation of motion for conservative system, where $f = T - V$ is function of $q_j, \dot{q}_j$ and t i.e.

$$f = f\left(q_j, \dot{q}_j, t\right)$$

Case II. Non-conservative system : For non-conservative system, the potentials are velocity depndent,

i. e., $V = V\left(q_j, \dot{q}_j\right)$ [q_j is generalised coordinate and $\dot{q}_j$ is generated velocity]

so, $Q_j = -\frac{\partial V}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_j}\right)$ ....(6.5.10)

Putting the equation (6.5.10) in equation (6.5.8) we obtain,

$$\sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial \dot{q}_j} \right] = \sum_j - \frac{\partial V}{\partial q_j} + \sum_j \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_j} \right)$$

or, $\sum_j \frac{d}{dt} \frac{\partial (T-V)}{\partial \dot{q}_j} - \sum_j \frac{\partial (T-V)}{\partial q_j} = 0$

or, $\sum_j \left[\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_j} \right) - \frac{\partial f}{\partial q_j} \right] = 0$, [as $L = T - V$]

The above equation is exactly same as equation (6.5.9). Hence we can say the Euler-Language equation is in the same form for conservative and non-conservative system.

when $f = T - V = L$, then equation (6.5.8) and (6.5.9) becomes

$$\sum_j \left[\frac{d}{dt}\left(\frac{\delta L}{\delta q_j}\right) - \frac{\delta L}{\delta q_j}\right] = 0 \quad \text{... (6.5.10) where L recognised as Lagrangian}$$

The lagrangian function is useful to fix the equation of motion. If we want to know the information about the path adopted by the system during the motion, it is necessary to provide six initial values for a particle-three for position co-ordinates and three for velocity co-ordinates. But in configuration space, specification of the position of a single point provides only three initial values for each particle and hence specification of path is not possible. Thus if we difine the equation of motion, there should be infinite number of possible paths through any print in configuration space. Configuration space is introduced to represent the motion of a system in Lagrangion approach.

6.5.2 Application of Lagrange's equation of motion :

1. One dimensional simple harmonic oscillator :

For simple harmonic oscillator, the kinetic energy of this system is

$$T = \frac{1}{2} m \dot{x}^2$$

and potential energy $V = -\int F\, dx$

$$= \quad -\int (-k\,x)\, dx \quad \text{[here } k \text{ is constant]}$$

$$= \quad +\frac{1}{2} kx^2 + c$$

Hence, Lagrangian L can be expressed in the form

$$L = T - V = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 - c$$

From equations (6.5.9) we get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \quad \text{[Here we put } q_j = x\text{]}$$

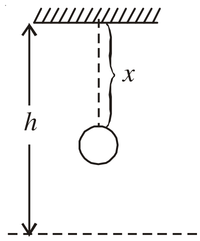
or, $m\ddot{x}+kx=0$

or, $\boxed{\ddot{x}+\omega^2 x=0}$ $\left[\text{where } w=\sqrt{\frac{k}{m}}\right]$

It is an equation of simple harmonic oscillator, where w is the frequency of oscillation given by $\omega=\sqrt{\frac{k}{m}}$.

2. Falling body in uniform gravity :

When an object falls from rest, its gravitational potential energy is converted to kinetic energy.



Hence, kinetic energy $T=\frac{1}{2}m\dot{x}^2$ [at any distance x]

and potential energy $V = mg\ (h - x)$

So, Lagrangian $L = T - V$

$$= \frac{1}{2}mx^2 - mg(h-x)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0$$

or, $m\ddot{x} - mg = 0$

or, $\boxed{m\ddot{x}=mg}$

For free falling body the above equation represents the equation of motion and acceleration is g *i*.e., acceleration due to gravity.

3. Simple pendulum :

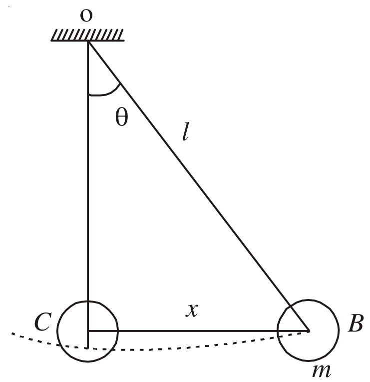
The angle θ between rest position and deflected position is chosen as generalised co-ordinate. If the length of the pendulum of mass m is l, then kinetic energy $T=\frac{1}{2}m\dot{x}^2$

$=\frac{1}{2}ml^2\dot{\theta}^2$

as potential energy $V = mg\ (l - l \cos \theta)$

Thus Lagrangian is $L = T - V$

$$= \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$$



Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$$

or, $ml^2\ddot{\theta} + mgl\sin\theta = 0$

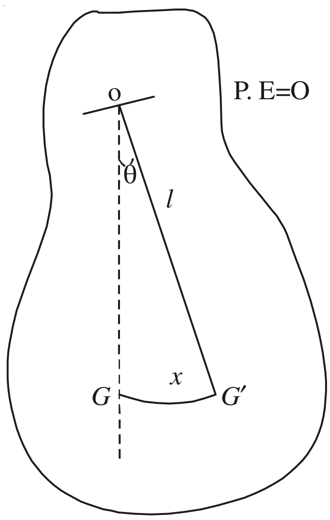
$$\ddot{\theta} + \frac{m}{l}\sin\theta = 0$$

For small θ, $\sin\theta = \theta$

So, $\boxed{\ddot{\theta} + \frac{m}{l}\theta = 0}$

The above equation represents the eqnation of motion of a simple pendulum with frequency

$\omega = \sqrt{\frac{m}{l}}$ and time period $T = 2\pi\sqrt{\frac{l}{g}}$.



4. Compound Pendulum : Let the point θ be fixed and taken it as origin. The point G is the centre of gravity in equilibrium position and G′, the centre of gravity in displaced position.

Kinetic energy $T = \frac{1}{2}I\dot{\theta}^2$

I is the moment of inertia of the body about the axis of rotation.

Potential energy, $V = -\ mgl \cos\theta$

$\therefore$ Lagrangian $L = T - V = \frac{1}{2}I\dot{\theta}^2 + mgl\cos\theta$

$$\therefore\ \frac{\partial L}{\partial \theta} = \dot{I}\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -mgl\ \sin\theta = -\,mgl\ \theta$$

We have, $\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

or, $I\ddot{\theta} + mgl\,\theta = 0$

$$\ddot{\theta} + \frac{mgl}{I}\theta = 0$$

or, $\ddot{\theta} + \omega^2\theta = 0$, where $\omega = \sqrt{\frac{mgl}{I}}$

Hence, time period $T = 2\pi\sqrt{\frac{I}{mgl}} = 2\pi\sqrt{\frac{k^2 + l^2}{gl}}$.

6.6 Concept of symmetry :

If the property of a system does not charge under some defined operations, the system is said to have a symmetry with respect to the given operation. A system which does not interact with particles or field outside the system may exhibit space homogeneity and isotropy. The conservation theorems to be discussed are first integrals of equation of motion. These are the relations of the type $f(q_j,\ \dot{q}_j,\ t)$ = constant.

6.6.1 : Cyclic or Ignorable co-ordinates :

We know that Lagrangian is a function of generalised co-ordinate q_j, generalised velocity $\dot{q}_j$, and time t, i.e.,

$$L = L\left(q_j, \dot{q}_j, t\right) \ldots \quad (6.6.1)$$

If Lagrangian does not contain any coordinate q_k from any set of co-ordinates that means L is independent of q_k or $\frac{\partial L}{\partial q_k} = 0$, then such co-ordinate is reffered to as an ignorable or cyclic co-ordinate.

6.6.2 Homogeneity of space :

Homogencity of space implies that the physical laws are invariant under space translation. If we change the position of each particles by some vector $\vec{dr}$ the Lagrangian of our system will be remain same, it implies the position vector

$\vec{r_j} \rightarrow \vec{r_j'} = \vec{r_j} + \vec{dr}$ and change of Lagrangian of

our system $\delta L = \sum_j \frac{\partial L}{\partial \vec{r_j}} \cdot \delta \vec{r_j} = \sum_j \frac{\partial L}{\partial \vec{r_j}} \cdot \delta \vec{r}$

Symmetry under orbitrary tanslation implies

$$\delta L = 0 \Rightarrow \sum_j \frac{\partial L}{\partial \vec{r_j}} = 0$$

Using Euler Language equaton, we can say

$$\sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \vec{r_j}}\right) = 0$$

Put, $L = \frac{1}{2} m_j \dot{r}_j^2 - v(r_i)$

then, $\frac{d}{dt}\left(\frac{\partial}{\partial \dot{\vec{r_j}}}\left(\frac{1}{2} m_j r_j^2\right)\right) = 0$

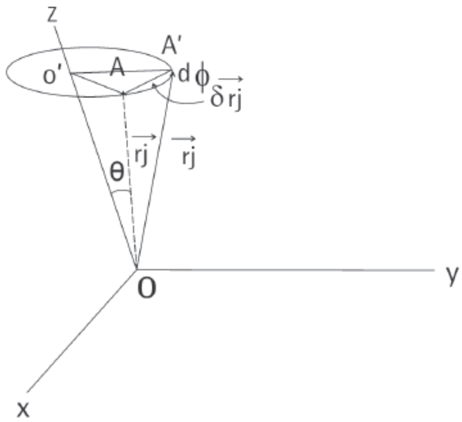
$$\text{or,} \quad \frac{d}{dt}\left(m_j \cdot \dot{\vec{r_j}}\right) = 0 \quad \dot{\vec{p_j}} = 0$$

i.e., total momentum of the system is constant. Thus homogeneity of space leads to conservation of Linear momentum of the system.

6.6.3 Isotropy of space :

Isotropy means rotation invariance in free space.

If Lagrangian L is independent of the orientation of the system, it results the conservation of angular momentum of the system. Let us consider the rotation about the z-axis by an angle θ so that OA → OA′.



It follows that $\vec{OA}' = \vec{OA} + \vec{AA}'$, where the magnitute $\vec{AA}'$ is given by $\left(\delta\vec{r_j}\right)$

$$\left(\vec{AA}'\right) = r_j \ \sin\theta \ d\phi$$

Thus $\vec{r_j} \to \vec{r_j}' = \vec{r_j} + \delta\vec{r_j} = r_j \sin\theta \, d\theta \ = \ d\vec{\phi} \times \vec{r_j}$

then $\vec{v_j} \to \vec{v_j} = \vec{v_j} + \delta\vec{v_j}$

where $\delta\vec{v_j} = d\vec{\phi} \times \vec{v_j}$

If the Lagrangian is invariant under rotation then we have,

$$\delta L = \sum_j \frac{\partial L}{\partial \vec{r}_j} \cdot \delta \vec{r}_j + \sum_j \frac{\partial L}{\partial \dot{\vec{r}}_j} \cdot \delta \dot{\vec{r}}_j$$

$$= \sum_j \frac{\partial}{dt}\left(\frac{\partial L}{\partial \vec{r}_j}\right) \cdot \delta \vec{r}_j + \sum_j \frac{\partial L}{\partial \dot{\vec{r}}_j} \cdot \delta \dot{\vec{r}}_j$$

$$= \sum_j \frac{\partial}{dt}\left(\frac{\partial L}{\partial \vec{r}_j}\right) \cdot \left(d\vec{\phi} \times \vec{r}_j\right) + \sum_j \frac{\partial L}{\delta \dot{\vec{r}}_j}\left(d\vec{\phi} \times \vec{v}_j\right)$$

Now, $L = \sum_j \frac{1}{2} m \dot{\vec{r}}_j^{\,2} - V\left(\vec{r}_j\right)$

Hence, $\delta L = \sum_j m \ddot{\vec{r}}_j \cdot \left(d\vec{\phi} \times \vec{d}_j\right) + \sum_j m \dot{\vec{r}}_j \left(d\vec{\phi} \times \dot{\vec{r}}_j\right)$

$$= \sum_j \dot{\vec{P}}_j \cdot \left(d\vec{\phi} \times \vec{r}_j\right) + \sum_j \vec{P}_j \cdot \left(d\vec{\phi} \times \dot{\vec{r}}_j\right)$$

$$= d\vec{\phi} \cdot \sum_i \frac{d}{dt}\left(\vec{r}_j \times \vec{P}_j\right)$$

If $\partial L = 0$, then $\sum_j \frac{d}{dt} \cdot \left(\vec{r}_j \times \vec{P}_j\right) = 0 \Rightarrow \vec{r}_j \times \vec{P}_j =$ constant

so the angular momentum of the system is constant.

6.6.4 Conservation of Linear momentum :

If a co-ordinate corresponding to a displacement is cyclic translation of the system has no effect i.e, description of system motion remians invariant under such a translation and linear momentum is conserved.

Let, q_j is the generalised co-ordinate and corresponding generalised displacement is $dq_{j.}$ From equation (6.5.9) we obtain.

$$\sum_j \frac{d}{dt}\left(\frac{dL}{d\dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=0$$

We know, $L = T - V$, where T is independent of q_j i.e.,

$$\sum_j \frac{d}{dt}\left(\frac{dL}{d\dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=0 \qquad ...(6.6.2)$$

Further Kinetic energy $T=\frac{1}{2}m\dot{r}_i^2$

$$\text{or, } \frac{\partial T}{\partial \dot{q}_j}=m\dot{\vec{r}}_i\cdot\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}$$

$$= m\dot{\vec{r}}_j\cdot\frac{\partial \dot{\vec{r}}_j}{\partial \dot{q}_j}$$

$$= \dot{\vec{P}}_i\cdot\frac{\partial \dot{\vec{r}}_i}{\partial q_j} \quad ... (6.6.3)$$

If $\hat{n}$ is the unit vector along the direction of translation, then

$$\partial\vec{r}_i = \hat{n}\,\partial q_j$$

$$\text{or, } \frac{\partial \vec{r}_i}{\partial q_j}=\hat{n}$$

Hence, $T = \left(\vec{P_i}\cdot\hat{n}\right)$

where, $\vec{P_i}$ represents the component of total linear momentum along the direction of translation.

So from eqation (6.6.2) we can say

$$\frac{d}{dt}\left(\vec{P_i}\cdot\hat{n}\right) = \frac{\partial L}{\partial q_j}$$

If q_j is cyclic, then $\frac{\partial L}{\partial q_j} = 0$ i.e., L is indepdendent of q_j.

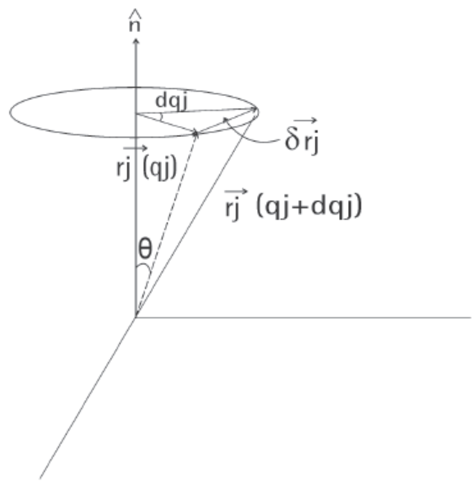
Hence $\left(\vec{P_i}\,.\hat{n}\right)$ = constant... (6.6.4)

Linear momentum is conserved if a co-ordinate corresponding to a displacement is cyclic.

6.6.5 Conservation of angular momentum :

If a co-ordinate corresponding to a rotation is cycdic, then system remain invariant under such a coordinate rotation and angular momentum is conserved.

The figure shows the change of position vector under rotation of the system. Here chang in q_j must correspond to an infinitesimal rotation of the vector $\vec{r_i}$.



$\vec{r_i}(q_j) \rightarrow \vec{r_i}(q_j + dq_j)$ after rotation of $\vec{r_i}$ i.e.

$d\vec{r_i} = \vec{r_i}(q_j + dq_j) \rightarrow \vec{r_i}(q_j)$

The magnetitude of $d\vec{r_i}$ is $\left(d\vec{r_i}\right) = r_1 \sin\theta \, dq_j$

$$\frac{\partial \vec{r_i}}{\partial q_j} = \hat{n} \times \vec{r_i}$$

From equation (6.6.3) we get,

$$\sum \frac{\partial T}{\partial \dot{q}_j} = \sum \vec{P_i} \cdot \frac{\partial \vec{r_i}}{\partial q_j} = \sum \vec{P_i} \cdot \left(\hat{n} \times \vec{r_i} \right) = \sum \vec{n} \cdot \left(\vec{r_i} \times \vec{P_i} \right) = \hat{n} \cdot \vec{L}$$

Hence $\sum \frac{\partial T}{\partial \dot{q}_j}$ represent the total angular momentum along the axis of rotation.

Putting the value of $\sum \frac{\partial T}{\partial \dot{q}_j}$ in equation (6.6.2) we get,

$$\frac{d}{dt}\left(\hat{n} \cdot \vec{L} \right) = \sum_j \frac{\partial L}{\partial q_j}$$

If q_j is cyclic, then $\frac{\partial L}{\partial q_j} = 0$, So,

$$\frac{d}{dt}\left(\hat{n} \cdot \vec{L} \right) = 0$$

or, $\hat{n} \cdot \vec{L}$ = constant. ... (6.6.5)

If the rotation coordiante is cyclic, then angular momentum along the rotation is conserved.

6.6.6 Conservation of energy :

For a conservative system, potential energy is a functin of coordinate not the velocity and constraints do not change with time. So, time will not involve explicity and hence

L can be written as $L(q_j, \dot{q}_j)$.

Thus total time derivative can be written as

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j$$

$$= \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j \quad \left[\because \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j}\right]$$

$$= \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right)$$

$$= \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial T}{\partial \dot{q}_j}\right) = 0 \quad [\text{ as } V \text{ is independ of } \dot{q}_j\,]$$

$$= \sum_j \frac{d}{dt}\left(\dot{q}_j P_j\right)$$

Thus $\frac{dL}{dt} - \sum_j \frac{d}{dt}\left(\dot{q}_j\ P_j\right) = 0$

or, $\frac{d}{dt}\left(\sum \dot{q}_i P_i - L\right) = 0$

or, $\sum \dot{q}_j P_j - L =$ constant ... (6.6.6)

We know, $P_j = m\dot{q}_j$

Hence, for simple consideration we can write

$$\dot{q}_j P_j = m_j \dot{q}_j^2 = 2.\left(\frac{1}{2} m_j \dot{q}_j{}^2\right) = 2T$$

Now, equation (6.6.6) becomes

$$2T - T + V = \text{constant}$$

or, $T + V =$ constant ... (6.6.7)

The above equation shows that sum of kinetic energy and potential energy *i.e.*, total energy of a system is invariant if time t does not occur in L explicitly j means t is cyclic.

6.7 Hamiltonian formulation of Mechanics

In Lagrangian formulation independent variables are generalised co-ordinates and time. As generated velocity is simply the time derivative of generalised coordinate, so generalised velocity is not the independent variable. In Hamiltonian formulation just we introduce a new independent variable, called generalised momentum P_j. Like Lagrangian, a new function of this formalism is Hamiltonism H, which is function of generalised coordinates, generalised momenta and time *i.e.*,

$$H = H\left(q_j, P_j, t\right).$$

Generalised momenta already have been defined by $P_j = \dfrac{\partial L}{\partial \dot{q}_j}$,

Which shows that, for each generalised coordinate, there is one component of generalised momentum. After providing the equal status of generalised momenta, the path adopted by the system during its motion must now be represented by a space of 6N dimensions (instead of 3N dimension) in which 3N dimensions for generalised co-ordinate and 3N dimensions for generalised momenta. This new space is called

phase space. In configuration space there are infinite number of possible paths during motion, which we already discussed. But in this new formulation (i.e., Hamiltonian), specification of initial values of q_j and P_j at any instant on the path will fix the whole path of the moving system. Hence path in phase space refers almost the actual dynamical path, which is almost impossible to get in configuration space.

6.7.1 Hamiltonian :

From equation (6.6.6) we obtain,

$$\sum_j \dot{q}_j P_j - L = \text{constant}$$

Now this constant is designated by a letter H. Thus we can write

$$\sum_j \dot{q}_j P_j - L = H$$

We recognise H as Hamiltonian and assign to it a basis of (q_j , P_j) set, i.e.,

$$H = H\left(q_j,\, P_i\right) = \sum \dot{q}_j P_j - L\left(q_j, \dot{q}_j\right) \qquad \text{... (6.7.1)}$$

If H does not involve time, it is said to be a constant of motion, and represents the total energy of the system as discussed earlier. It is possible that H may be a constant of motion but not the total energy of the system.

6.7.2 Variational principle :

The principle of least action or more accurately,the principle of stationary action— is a variational principle that, when applied to the action of a mechanical system, can be used to obtain the equation of motion for that system. The principle is called "least" because its solution requires finding the path that has the least value. The principle can be used to derive Newtonian, Lagrangian and Hamiltonian equations of motion.

Let f be a function of many independent variables q_j and their derivatives $\dot{q}_j$. Then integral I, representing a path between two points 1 and 2 can be written as

$$I = \int_1^2 f\left(q_j(x), \dot{q}_j(x), x\right) dx$$

Mathematicaly the principle states that

$\delta I = 0,$

In words the path taken by the system between 1 and 2 and configurations from points 1 to 2 is the one for which the action is stationary (no change) to first order.

6.6.3 Hamilton's principle :

Hamilton's variational principle or Hamilton's principle inovel with motion of the system. This principle states that the integral $\int_{t_1}^{t_2}(T-V)\,dt$ have a stationary value (extremum) t_1 where T is the kinetic energy and V is the potential energy of the system.

T is a function of cordinates and their derivatives and V is a function of co-ordinates only. Hence Hamilton's principle is only for conservative system and state that the motion of the system from time t_1 to time t_2 is such that lime integral

$$I = \int_{t_1}^{t_2} L\,dt, \text{ where } L = T - V \quad ....(6.7.2)$$

is the extremum for the path of motion, is defined as Lagrangian (already discussed earlier).

6.7.4 Derivation of Lagrange's equation :

Let us consider a conservative system of particles. In the form of generalised coordinates equation (6.7.2) can be written as

$$I = \int_{t_1}^{t_2} L\left(q_j, \dot{q}_j\right) dt$$

$$= \int_{t_1}^{t_2} \left[T\left(q_j, \dot{q}_j\right) - V\left(q_j\right)\right] dt$$

According to Hamilton's principles, we have

$$\delta I = 0 \text{ (Entremum condition)}$$

or, $\delta\int_{t_1}^{t_2}\left[T\left(q_j,\dot{q}_j\right)-V\left(q_j\right)\right]dt=0$

or, $\int_{t_1}^{t_2}\sum_j\left[\left(\frac{\partial T}{\partial q_j}\,\delta q_j+\frac{\partial T}{\partial \dot{q}_j}\delta\dot{q}_j\right)-\frac{\partial V}{\partial q_j}\delta q_j\right]dt=0$

or, $\int_{t_1}^{t_2}\sum_j\frac{\partial(T-V)}{\partial q_j}\delta q_j dt+\int_{t_1}^{t_2}\sum_j\frac{\partial T}{\partial \dot{q}_j}\delta\dot{q}_j\,dt=0$

or, $\int_{t_1}^{t_2}\sum_j\frac{\partial(T-V)}{\partial q_j}\delta q_j dt+\int_{t_1}^{t_2}\frac{d}{dt}\left(\sum_j\frac{\partial T}{\partial \dot{q}_j}\delta q_j\right)dt\;-\int_{t_1}^{t_2}\frac{d}{dt}\left(\sum_j\frac{\partial t}{\partial \dot{q}_j}\right)\partial q_j\;dt=0$

or, $\int_{t_1}^{t_2}\sum_j\frac{\partial\left(T-V\right)}{\partial q_j}\partial q_j\;dt\;+\sum_j\frac{\partial T}{\partial \dot{q}_j}\delta q_j\Big|_{t_1}^{t_2}-\int_{t_1}^{t_2}\frac{d}{dt}\sum_j\left(\frac{\partial T}{\partial \dot{q}_j}\right)\delta q_j dt=0$

Since there is no co-ordiante variation at end points, so

$\delta q_j\Big|_{t_1}^{t_2}=0$ and above equation reduces to

$$\int_{t_1}^{t_2}\sum_j\frac{d(T-V)}{dq_j}\delta q_j.dt-\int_{t_1}^{t_2}\frac{d}{dt}\sum_j\left(\frac{\partial T}{dq_j}\right)\delta q_j dt=0$$

or, $\int_{t_1}^{t_2}\sum_j\left[\frac{d(T-V)}{dq_j}-\frac{d}{dt}\left(\frac{\partial T}{d\dot{q}_j}\right)\right]\delta q_j dt=0$

Since δq_j is independent to each other, the coefficent of every δq_j, is zero i.e.,

$$\frac{\partial (T-V)}{\partial q_j} - \frac{d}{dt}\left(\frac{dT}{d\dot{q}_j}\right) = 0$$

For conservative system, we can write

$$\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial \dot{q}_j} = 0 \quad \text{[As } V \text{ is independent of } \dot{q}_j \text{]}$$

or, $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_j}\right) - \dfrac{\partial L}{\partial \dot{q}_j} = 0$ [As $L = T - V$]

For set of coordinates in a configuration system the equation can be written as

$$\sum_j \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right] = 0 \qquad \text{... (6.7.3)}$$

This set of equations are called Lagrange's equation of motion. This equations follow directly from Hamilton's principle.

6.7.5 Hamilton's equations of motion :

If we consider the Hamiltonian as defined in equation (6.7.1) is also the function of time, then Hamiltonian (H) can be represented by

$$H = H\,(q_j, P_j, t) \qquad \text{... (6.7.4)}$$

and equation (6.7.1) becomes

$$H = \sum \dot{q}_j P_j - L\left(q_j, \dot{q}_j, t\right) \quad \text{[as } L = L\left(q_j, \dot{q}_j, t\right) \text{]} \qquad \text{... (6.7.5)}$$

The differential of eqn. (6.7.4) gives

$$dH = \sum_j \frac{\partial H}{\partial q_j} dq_j + \sum_j \frac{\partial H}{\partial P_j} dP_j + \frac{\partial H}{\partial t} dt \qquad \text{........(6.7.6)}$$

and the differential of equation (6.7.5) gives

$$dH = \sum_j \dot{q}_j dp_j + \sum_j P_j d\dot{q}_j - \sum_j \frac{\partial L}{\partial q_j} dq_j - \sum_j \frac{\partial L}{\partial t_j} d\dot{q}_j \; - \frac{\partial L}{\partial t} dt \qquad \text{... (6.7.7)}$$

we have, $\frac{\partial L}{\partial \dot{q}_j} = P_j$ and $\frac{\partial L}{\partial q_j} = P_j$

Putting this values in equation (6.7.7) we get,

$$dH = \sum_j \dot{q}_j dP_j + \sum_j P_j d\dot{q}_j - \sum_j \dot{P}_j dq_j - \sum_j P_j d\dot{q}_j - \frac{\partial L}{\partial t} dt$$

$$= \sum_j \dot{q}_j dP_j - \sum_j \dot{P}_j dq_j - \frac{\partial L}{\partial t} dt \qquad \text{... (6.7.8)}$$

Compairing the coefficients of equations (6.7.6) and (6.7.8) we get,

$$\left.\begin{aligned} \dot{q}_j &= \frac{\partial H}{\partial P_j} \\ \text{and} \quad \dot{P}_j &= -\frac{\partial H}{\partial q_j} \end{aligned}\right\} \qquad \text{... (6.7.9)}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \qquad \text{... (6.7.10)}$$

Equation (6.7.9) are known as Hamiltons canonical equations of motion. Generalised momenta and generalised coordinates are dynamically equivalent sets of variables, because their role can be interchanged just by making a change of sign.

Hence we must say that Hamilton's equation of motion are symmetric in q_j and P_j except for a change in sign.

6.7.6 Advantage of Hamiltonian Approach :

In Lagrangian approach, two variables q_j and $\dot{q}_j$ are not in equal status where as in Hamiltonian approach, co-ordinates and momenta are placed at equal footing, that provides the freedom of choosing sets of coordinates and momenta. The knowledge of Hamiltonian of a system is extremely important particualrly if we are interested in quantising a dynamical system, because equality of status of coordinates and momenta provides a convenient basis for the development of quantum mechanics.

6.7.7 Applications of Hamiltonian formulation :

1. Simple Harmonic oscillator :

For simple Harmonic oscillator, Kinetic energy $T = \frac{1}{2}m\dot{x}^2$ and potential energy $V = \frac{1}{2}kx^2$

Hence, Lagrangian $L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

$$P_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$\therefore$ Hamiltonian $H = P_x\dot{x} - L = m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

$$= \frac{P_x^{\,2}}{2m} + \frac{1}{2}kx^2$$

Canonical equations are $\dot{x} = \frac{\partial H}{\partial P_x} = \frac{P_x}{m}$

and $\dot{P}_x = -\frac{\partial H}{\partial P_x} = -kx$

or, $\boxed{m\ddot{x} + kx = 0}$,

This is equation of motion of simple Harmonic oscillator.

In two dimensions $T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$, $V = \frac{1}{2}k_1 x^2 + \frac{1}{2}k_2 y^2$

$$P_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, : P_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$$

$$\therefore \quad H = P_x \dot{x} + P_y \dot{y} - L$$

$$= \frac{P_x^2}{m} + \frac{P_y^2}{m} - 2m\left(P_x^2 + P_y^2\right) + \frac{1}{2}k_1 x^2 + \frac{1}{2}k_2 y^2$$

$$= \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + \frac{1}{2}k_1 x^2 + \frac{1}{2}k_2 y^2$$

Hence, Hamilton's Canonical equations are

$$\dot{x} = \frac{\partial H}{\partial P_x} = \frac{P_x}{m}; \qquad \dot{y} = \frac{\partial H}{\partial P_y} = \frac{P_y}{m};$$

$$\dot{p}_x = -\frac{\partial H}{\partial P_x} = -k_1 x; \quad \dot{p}_y = -\frac{\partial H}{\partial y} = -k_2 y$$

or, $\dot{p}_x = -k_1 x = -m\ddot{x}$; $\dot{p}_y = -k_2 y = m\ddot{y}$

$$\left.\begin{array}{ll} \therefore & m\ddot{x} + k_1 x = 0 \\ \text{and} & m\ddot{y} + k_2 y = 0 \end{array}\right\}$$

Proceeding similarly, for 3-dimensional relations we can write, $m\ddot{x} + k_1 x = 0$; $m\ddot{y} + k_2 y = 0$, $m\ddot{z} + k_3 z = 0$

2. Simple Pendulum :

We have $T = \frac{1}{2}ml^2\dot{\theta}^2$, $v = mgl\,(1 - \cos\theta)$

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)$$

Hence, $P_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$

Now, $H = P_\theta\dot{\theta} - L = ml^2\dot{\theta}^2 - \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta)$

$$= \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta)$$

$$= \frac{P_\theta{}^2}{2ml^2} + mgl(1-\cos\theta)$$

Hamilton's Canonical equation are

$$\dot{\theta} = \frac{\partial H}{\partial P_\theta} = \frac{P_\theta}{ml^2}$$

and $\dot{P}_\theta = -\frac{\partial H}{\partial \theta}$ = – mgl sin θ = –mglθ $[\because \sin\theta \approx \theta]$

$$\ddot{\theta} = \frac{\dot{P}_\theta}{ml^2} = -\frac{mgl\theta}{ml^2}$$ $m \quad r^2 \quad \dot{\theta}$

or, $\ddot{\theta} = -\frac{g}{l}\theta$

$$\boxed{\therefore\ \ddot{\theta} + \frac{g}{l}\theta = 0}$$

3. Particle in a central field of force :

For central field, force is always directed towards the centre and potential energy is only the function of coordinates i.e., system is conservative. As the motion always confined in a plane, we consider it in (r, θ) coordinates.

Now, Kinetic energy for the system is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

and $V = V(r)$

$\therefore$ Lagrangian, $L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)$ ... (1)

Momenta $P_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$ ... (2)

and $P_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$ ... (3)

For central force,

$$F(r) \propto -\frac{1}{r^2}$$

or, $\vec{F}(r) = -\frac{k}{r^2}\hat{r}$ [Gravitational force and Coulomb force are two examples]

We put, $-\frac{dV(r)}{dr} = F(r) = -k/r^2$

or, $V(r) = \int \frac{k}{r^2} dr + c = -\frac{k}{r} + c$

for $r \to \infty, v(r) \to 0 \therefore c = 0$

So, $V(r) = -\frac{k}{r}$ ... (4)

Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

or, $\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$ [Putting $L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)$

or, $mr^2\dot{\theta}$ = constant = l (say)

Hence, $P_\theta = mr^2\dot{\theta} = l$ [From equation (3)]

or, $\dot{\theta} = \frac{l}{mr^2}$

Hamiltonian

$$H = P_r \dot{r} + P_\theta \dot{\theta} - L$$

$$= \frac{P_r^2}{m} + \frac{P_\theta^2}{mr^2} - \frac{P^2 r}{2m} - \frac{P_\theta^2}{2mr^2} + V(r)$$

[Putting the values of equation (1), (2) and (3)]

$$= \frac{P_r^2}{2m} + \frac{P_\theta^2}{2mr^2} - \frac{K}{r} \quad \text{[From equation (4)]}$$

..... (5)

Hamiltons's canonical equation are

$$\left.\begin{aligned} \dot{r} &= \frac{\partial H}{\partial P_r} = \frac{P_r}{m} \\ \text{and} \quad \dot{\theta} &= \frac{\partial H}{\partial P_\theta} = \frac{P_\theta}{mr^2} \end{aligned}\right\} \quad \text{... (6)}$$

$$\left.\begin{aligned} \dot{P}_r &= -\frac{\partial H}{\partial r} = \frac{P_\theta{}^2}{mr^3} - \frac{k}{r^2} \\ \text{or,} \quad \dot{P}_\theta &= -\frac{\partial H}{\partial \theta} = 0 \end{aligned}\right\} \quad \text{... (7)}$$

From equation (6) we get

$$\dot{r} = \frac{P_r}{m}$$

$$\text{or,} \quad \ddot{r} = \frac{\dot{P}_r}{m} = \frac{P_\theta{}^2}{m^2 r^3} - \frac{k}{mr^2} \quad \text{[from equation (7)]}$$

$$\text{or,} \quad \boxed{m\ddot{r} = \frac{P_\theta{}^2}{mr^3} - \frac{k}{r^2}} \quad \text{... (8)}$$

4. Hamiltonian for a free particle

For a free particle, the potential energy is constant, and may be taken as zero. The kinetic energy

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

$\therefore$ Lagrangian L = T − V

$$= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

$$P_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \dot{x} = \frac{P_x}{m}$$

$$P_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \Rightarrow \dot{y} = \frac{P_y}{m}$$

$$P_z = \frac{\partial L}{\partial \dot{y}} = m\dot{z} \Rightarrow \dot{z} = \frac{P_z}{m}$$

$\therefore$ Hamiltonian $H = P_x\dot{x} + P_y\dot{y} + P_z\dot{z} - L$

$$= \frac{P_x^2}{m} + \frac{P_y^2}{m} + \frac{P_z^2}{m} - \frac{1}{2}m\left(\frac{P_x^2}{m^2} + \frac{P_y^2}{n^2} + \frac{P_z^2}{m}\right)$$

$$\therefore \quad H = \frac{1}{2m}\left(P_x^2 + P_y^2 + P_z^2\right)$$

6.8 Summary :

1. Constraints is defined with some examples.
2. Degrees of freedom is defined with some examples.
3. Generalised coordinates, generalised displacement, virtual work, D' Alembert's principle have been discussed.
4. Euler–Lagrange's equation, and its applications have been discussed.
5. For conservative system, potential function is only dependent on generalised coordinates whereas for non-conservative system the potentials are velocity dependent.
6. Concept of symmetry, homogeneity of space, isotropy of space, different conservation laws, Hamiltonian formulation, applications of Hamiltonian formulation have been discussed.

6.9 Review Questions and Answer :

1. What are generalised coordinates? What is the advantage of using them?

Ans. See the article no. (6.4).

2. What are constraints? Explain with examples.

Ans. See the article no. (6.2).

3. It is not necessary for generalised force Q_j to have the dimensions of force but it is necessary that the product Q_j dq_j must have the dimensions of work. Justify.

Ans. We have from equation (6.4.3), the work done

$$\delta W = \sum_i Q_j \delta q_j \quad \text{where} \quad Q_j = \sum_{i=1}^{N} \vec{F_i} \cdot \frac{\partial \vec{r_i}}{\partial q_j}$$

If we consider a particle defined by (r, θ) genaralised coordinat, the component of forces are

$$\vec{F} = F_r \hat{r} + F_\theta \hat{\theta}$$

The generlised force for r-coordinate

$$Q_r = \vec{F} \cdot \frac{\partial \vec{r}}{\partial r} = \left(F_r \hat{r} + F_\theta \hat{\theta}\right) \cdot \frac{\partial r}{\partial r} \hat{r} = F_r$$

Have the dimension for force

But, the generalised force for θ-coordinate

$$Q_\theta = \vec{F} \cdot \frac{\partial \vec{r}}{\partial \theta} = \left(F_r \hat{r} + F_\theta \hat{\theta}\right) \cdot \left(r \cdot \frac{\partial r}{\partial \theta} \hat{r}\right)$$

$= r\, F_\theta$, have the dimension of torque. Hence the 1st statement is justified.

Now, work done for Q_θ is = $Q_\theta \theta$ = $rF_\theta \theta$ and for θ_r is = rF_r,

Both have the dimension of work, hence 2nd statement is justified.

4. Obtain an expression for generalised acceleration.

Ans. We have from equation (6.4.1)

$$\vec{r_i} = \vec{r_i}\left(q_1, q_2, \ldots\ldots\ldots\ldots q_n, t\right)$$

So, $$\dot{\vec{r_i}} = \frac{\partial \vec{r_i}}{\partial t} = \sum_{j=1}^{n} \frac{\partial \vec{r_i}}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r_i}}{\partial t}$$

Here, $\dot{q}_j$ is the time derivative of generalised coordinate q_j, is called the generalised velocity corresponding to the coordinate q_j.

Now, $$\ddot{\vec{r}_i} = \frac{d}{dt}\left\{\sum_{j=1}^{n}\frac{\partial \vec{r}_i}{\partial q_i}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}\right\}$$

$$= \sum_{j=1}^{n}\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_i}\right)\dot{q}_j + \sum_{j=1}^{n}\frac{\partial \vec{r}_i}{\partial q_i}\ddot{q}_j + \frac{\partial \dot{\vec{r}_i}}{\partial t}$$

$$= \sum_{j=1}^{n}\frac{\partial \dot{\vec{r}_i}}{\partial q_i}\dot{q}_j + \sum_{j=1}^{n}\frac{\partial \vec{r}_i}{\partial q_i}\ddot{q}_j + \frac{\partial \dot{\vec{r}_i}}{\partial t}$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 \vec{r}_i}{\partial q_i \partial q_k}\dot{q}_j\dot{q}_k + \sum_{j=1}^{n}\frac{\partial^2 \vec{r}_i}{\partial q_i \cdot \partial t}\dot{q}_j$$

$$+ \sum_{j=1}^{n}\frac{\partial^2 \vec{r}_i}{\partial q_j}\ddot{q}_j + \sum_{j=1}^{n}\frac{\partial^2 \vec{r}_i}{\partial q_j dt}\dot{q}_j + \sum_{j=1}^{n}\frac{\partial \vec{r}_i}{\partial q_i}\ddot{q}_j + \frac{\partial^2 \vec{r}_i}{\partial t^2}$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 \vec{r}_i}{\partial q_i \partial q_k}\dot{q}_j\dot{q}_k + 2\sum_{j=1}^{n}\frac{\partial^2 \vec{r}_i}{\partial q_j \cdot \partial t}\dot{q}_j + 2\sum_{j=1}^{n}\frac{\partial \vec{r}_i}{\partial q_j}\ddot{q}_j + \frac{\partial^2 \vec{r}_i}{\partial t^2}$$

The above expression contains the terms $\ddot{q}_j$, which is called generalised acceleration corresponding to the coordinate q_j

5. Prove the laws of conservation of Linear momentum. angular momentum and energy of a system in configuration space.

Ans. See article no. (6.6)

6. Derive Euler Lagrange equations of motion of a system using generatised coordinates.

Ans. Let *f* be *a* function of generalised coordinate q_j ; and their derivative $\dot{q}_j$; then path integral between two points an be written as

$$I = \int_{x_1}^{x_2} f\left(q(x), \dot{q}_j(x), x\right) dx \quad \text{[where } x \text{ is the independent variable]}$$

For a small variation of this integral I, we can write

$$\delta I = \int_{x_1}^{x_2} \sum_j \left[\frac{\partial f}{\partial q_j} \delta q_j + \frac{\partial f}{\partial \dot{q}_j} \delta \dot{q}_j \right] dx$$

$$= \int_{x_1}^{x_2} \sum_j \frac{\partial f}{\partial q_j} \partial q_j + \int_{x_1}^{x_2} \sum_j \frac{d}{dx}\left[\left(\frac{\partial f}{\partial \dot{q}_j}\right) \delta q_j\right] dx \; - \int_{x_1}^{x_2} \sum_j \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{q}_j}\right) \delta q_j dx$$

$$= \int_{x_1}^{x_2} \sum_j \frac{\partial f}{\partial q_j} \delta q_j dx - \int_{x_1}^{x_2} \sum_j \frac{d}{dx}\left(\frac{df}{d\dot{q}_j}\right) \delta q_j dx + \sum_j \frac{\partial f}{\partial \dot{q}_j} \delta q_j \Big|_{x_1}^{x_2}$$

At the end points [x_1 and x_2], all paths meet, so 3rd term should be zero,.

$$= \int_{x_1}^{x_2} \left[\sum_j \frac{\partial f}{\partial q_j} - \sum_j \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{q}_j}\right) \right] \delta q_i \, dx$$

For integral *I*, to be extrimum,

δI = 0

$$= \sum_j \frac{df}{dq_j} - \frac{d}{dx}\left(\frac{df}{d\dot{q}_j}\right) \Bigg] = 0$$

$$\frac{\partial f}{\partial q_j} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{q}_j}\right) = 0$$

or, $$\boxed{\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{q}_j}\right)-\frac{\partial f}{\partial q_j}-=0}$$

This is the expression of Euler Language equation in terms of generalised coordinates.

7. State Hamilton's principle and derive Lagrange eqnation of motion from it.

Ans. See articles (6.7.3) and (6.7.4).

8. Derive the Lagrangian for a charged particle in an electromagnetic field.

Ans. In an electromagnetic field, the force on a charged particle is given by

$$\vec{F}=q\left[\vec{E}+\left(\vec{v}\times\vec{B}\right)\right] \qquad \text{... (1)}$$

$\vec{F}$ can be expressed in terms of vector and scalar potential $\vec{A}$ and ϕ, and is given by

$$\vec{F}=q\left[-\vec{\nabla}\phi-\frac{\partial\vec{A}}{\partial t}+\vec{v}\times\left(\vec{\nabla}\times\vec{A}\right)\right] \quad \ldots \qquad (2)$$

where, $\vec{B}=\vec{\nabla}\times\vec{A}$

and $\vec{E}=-\vec{\nabla}\phi-\frac{\partial\vec{A}}{\partial t}$

Now, $$\left(\vec{\nabla}\phi\right)_x=\frac{\partial\phi}{\partial x}$$

and $$\left[\vec{v}\times\left(\vec{\nabla}\times\vec{A}\right)\right]_x=\frac{\partial}{\partial x}\left(\vec{v}\cdot\vec{A}\right)-\left[v_x\frac{\partial A_x}{\partial x}+v_y\frac{\partial A_x}{\partial y}+v_z\frac{\partial A_x}{\partial z}\right]$$

Total time derivative of A_x is

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \left(v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_x}{\partial y} + v_z \frac{\partial A_x}{\partial z} \right)$$

Hence, $\left(\vec{v} \times \vec{\nabla} \times \vec{A} \right)_x = \frac{\partial}{\partial x}\left(\vec{v} \cdot \vec{A} \right) - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t}$

The x-component of equation (2) is

$$F_x = q\left[-\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t} + \frac{\partial}{\partial x}\left(\vec{v} \cdot \vec{A} \right) - \frac{dA_x}{dt} + \frac{\partial A_x}{dt} \right]$$

$$= q\left[-\frac{d}{dx}\left(\varphi - \vec{v}.\vec{A} \right) - \frac{dA_x}{dt} \right]$$

Now, $\frac{\partial}{\partial v_x}\left(\vec{v} \cdot \vec{A} \right) = A_x$, Since $\frac{\partial A_x}{\partial v_x} = 0$

$$= q\left[-\frac{\partial}{\partial x}\left(\phi - \vec{v} \cdot \vec{A} \right) - \frac{d}{dt}\frac{\partial}{\partial v_x}\left(\vec{v} \cdot \vec{A} \right) \right]$$

As ϕ is independent of $v_{x,}$ Hence we

can write $F_x = q\left[-\frac{\partial}{\partial x}\left(\phi - \vec{v} . \vec{A} \right) + \frac{d}{dt}\frac{\partial}{\partial v_x}\left(\phi - \vec{v} . \vec{A} \right) \right]$

Let us put $V = q\left(\phi - \vec{v} \cdot \vec{A} \right)$, then $F_x = q\left[-\frac{\partial V}{\partial x} + \frac{d}{dt}\frac{\partial V}{\partial v_x} \right]$

Equation (6.5.8) gives us

$$\frac{d}{dt}\left(\frac{\partial T}{\partial v_x} \right) - \frac{\partial T}{\partial v_x} = F_x = -\left[\frac{\partial V}{\partial v_x} + \frac{d}{dt}\frac{\partial V}{\partial v_x} \right]$$

$$\text{or, } \frac{d}{dt}\left[\frac{\partial(T-V)}{\partial v_x}\right]-\left[\frac{\partial(T-V)}{\partial x}\right]=0$$

$$\text{or, } \frac{d}{dt}\left(\frac{\partial L}{\partial v_x}\right)-\frac{\partial L}{\partial x}=0$$

Where $L = T - V$ is the Lagrangian (as mentioned earlier)

$$\therefore\ L = T - V = T - q\phi + q\left(\vec{v}\,.\,\vec{A}\right)$$

The above expression is the Lagrangian for a charged particle in an electromagnetic field.

9. What is the advantage of Lagrangian approach over Newtonian approach?

Ans. (i) Lagrange's equations of motion are invariant in form, under the coordinate transformation.

(ii) Energy and work are more fundamentals than force, in the motion of a system. Hence Lagrangian approach is more advantageous than Newtonian approach.

(iii) In Lagrangian approach the choice of the coordinate is generalised, hence need not to tranform from centerian to polar, spherical, cylindrical etc.

10. Prove that shortest distance between two points in plane is straight line joining them using Hamilton's principle.

Ans. For an element of small arc length ds in a plane can be represented by

$$ds=\sqrt{dx^2+dy^2}=dx\sqrt{1+y^2}$$

Then, $I=\int_1^2 ds=\int_1^2\left(1+\dot{y}^2\right)^{\frac{1}{2}}dx=\int_1^2 f dx$

where $f=\sqrt{1+\dot{y}^2}$

For this curve to be extremum, δI = O and from Hamilton principle we get

$$\frac{\partial f}{\partial y}-\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \dot{y}}\right)=0 \qquad \text{[From equation. 6.6.3]}$$

$$\Rightarrow \frac{\partial f}{\partial y}=0$$

f = constant

or, $1+\dot{y}^2$ = constant

or, $\dot{y}$ = constant = m say

$\boxed{y=mx+c}$ [Integrating both sides]

This is the equation of a straight line.

11. State and explain the Hamiltons principle of least action? Derive Lagrange's equations from the Hamilton's principle.

Ans. See article no. (6.7.2) and (6.7.4)

12. Derive the canomnical equations of Hamilton.

Ans. See articles no. (6.7.5).

13. The Lagrangian is given by $L=-\left(1-\dot{q}^2\right)^{1/2}$. Write down the Lagrarge equation of motion.

Ans. we have,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}=0$$

$$\frac{dL}{d\dot{q}}=-\frac{1}{2}\left(1-\dot{q}^2\right)^{-\frac{1}{2}}-(-2\dot{q})$$

$$=\frac{\dot{q}}{\left(1-\dot{q}^2\right)^{\frac{1}{2}}}$$

and $\frac{dL}{dq} = 0$

Lagrangian equation is given by

$$\frac{d}{dt}\left[\frac{\dot{q}}{\left(1-\dot{q}^2\right)^{1/2}}\right] = 0$$

or, $\frac{\ddot{q}}{\left(1-\dot{q}^2\right)^{1/2}} - \frac{1}{2}\dot{q}\left(1-\dot{q}\right)^{-3/2} \cdot \left(-2\dot{q}\right)\ddot{q} = 0$

or, $\ddot{q}\left(1-\dot{q}^2\right) + \dot{q}^2\ddot{q} = 0$

or, $\ddot{q}\left(1-\dot{q}^2+\dot{q}^2\right) = 0$

or, $\ddot{q} = 0$

or, $\frac{dq}{dt} = m$ [Integrating both sides]

or, $\boxed{q = mt + c}$

14. The Lagrangian is given by $L(x,\dot{x}) = \frac{1}{2}\dot{x}^2 - \frac{1}{2}w^2x^2 - \alpha x^3 + \beta x\dot{x}^2$, where α, β, ω are constants. Find the Hamiltonian.

Ans. We have, $L(x,\dot{x}) = \frac{1}{2}\dot{x}^2 - \frac{1}{2}w^2x^2 - \alpha x^3 + \beta x\dot{x}^2$

$\therefore\ P_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} + 2\beta x\dot{x}$

$\Rightarrow\ \dot{x} = \frac{P_x}{1+2\beta x}$

$$\therefore\ H(x,\ P_x) = P_x\dot{x} - L$$

$$= P_x\dot{x} - \frac{1}{2}\dot{x}^2 + \frac{1}{2}w^2.x^2 + \alpha x^3 - \beta x\dot{x}^2$$

$$= \frac{P_x{}^2}{1+2\beta x} - \left(\beta x + \frac{1}{2}\right)\frac{P_x{}^2}{(1+2\beta x)^2} + \frac{1}{2}w^2x^2 + \alpha x^3$$

$$= P_x^2\left[\frac{1+2\beta x - \beta x - \frac{1}{2}}{(1+2\beta x)^2}\right] + \frac{1}{2}w^2x^2 + \alpha x^3$$

$$= \frac{P_x^2(2\beta x+1)}{2(1+2\beta x)^2} + \frac{1}{2}w^2x^2 + \alpha x^3$$

$$= \frac{P_x^2}{2(1+2\beta x)} + \frac{w^2x^2}{2} + \alpha x^3$$

15. The Lagrangian of a particle is given by $L = \frac{1}{5}\dot{q}^2 + \alpha(q-q_0)^2$, where α is constant. Derive the Hamiltonian.

Ans. We have, $L = \frac{1}{5}\dot{q}^2 + \alpha(q-q_0)^2$

$$P = \frac{\partial L}{\partial \dot{q}} = \frac{2}{5}\dot{q} \Rightarrow \dot{q} = \frac{5P}{2}$$

$\therefore$ Hamiltonian $H = p\dot{q} - L$

$$= \frac{5P^2}{2} - \frac{1}{5}\left(\frac{5P}{2}\right)^2 - \alpha(q-q_0)^2$$

$$= \frac{5P^2}{2} - \frac{5P^2}{4} - \alpha(q-q_0)^2$$

$$= \frac{5P^2}{4} - \alpha(q - q_0)^2$$

16. A Lagrangian is in the form $L = \frac{1}{2}\alpha\dot{q}^2 - \frac{1}{2}\beta q^2$, where α and β are constants. Find Hamiltonian and equation of motion.

Ans. We have, $L = \frac{1}{2}\alpha\dot{q}^2 - \frac{1}{2}\beta q^2$

$$P = \frac{\partial L}{\partial \dot{q}} = \alpha\dot{q} \Rightarrow \dot{q} = \frac{P}{\alpha}$$

$$\therefore \quad H = p\dot{q} - L$$

$$= \frac{P^2}{\alpha} - \frac{P^2}{2\alpha} + \frac{1}{2}\beta q^2$$

$$= \frac{P^2}{2\alpha} + \frac{1}{2}\beta q^2$$

Hamilton's canonical euations are

$$\dot{q} = \frac{\partial H}{\partial P} = \frac{P}{\alpha}$$

and $\dot{p} = -\frac{\partial H}{\partial q} = -\beta q$

or, $\alpha\ddot{q} + \beta q = 0$

or, $\boxed{\ddot{q} + \frac{\beta}{\alpha} q = 0}$

This is equation of motion of a simple harmonic motion.

17. The constraints given below is holonomic or not, explain.
xdx + ydy + 2zdz = 0

Ans. Yes, the constraint is holonomic i.e, integrable.

The integral relation is

$$\frac{x^2}{2}+\frac{y^2}{2}+z^2 = \text{constant [Integrating the equation]}$$

18. The kinetic and potential energy of a system is given by $T = A(\dot{q}_1^2 + q_1^2\dot{q}_2^2)$ and $V = f(q_1)$, respectively.

(a) What is the number of degrees of freedom of the system? (b) Write down the Hamiltonian of the system. (c) What is conserved quantity?

Ans. (a) There are two generalised coordinates q_1 and q_2. So the number of degrees of freedom is 2.

(b) The Lagrangian of the system is

$$L = T - V = A\left(\dot{q}_1^2 + q_1^2\dot{q}_2^2\right) - f(q_1)$$

Hence, $$P_1 = \frac{\partial L}{\partial \dot{q}_1} = 2A\dot{q}_1 \Rightarrow \dot{q}_1 = \frac{P_1}{2A}$$

$$P_2 = \frac{\partial L}{\partial \dot{q}_2} = 2Aq_1^2\dot{q}_2 \Rightarrow \dot{q}_2 = \frac{P_2}{2Aq_1^2}$$

Hamiltonian of the system is

$$H = P_1\dot{q}_1 + P_2\dot{q}_2 - L$$

$$= \frac{P_1^2}{2A} + \frac{P_2^2}{2Aq_1^2} - A\left[\frac{P_1^2}{4A^2} - q_1^2\frac{P_2^2}{4A^2q_1^4}\right] + f(q_1)$$

$$= \frac{P_1^2}{4A} + \frac{P_2^2}{4Aq_1^2} + f(q_1)$$

(c) Now, Hamilton equation gives us

$$\dot{P}_1 = -\frac{\partial H}{\partial q_1} \text{ and } \dot{P}_2 = -\frac{\partial H}{\partial q_2}$$

$$\dot{P}_2 = O \Rightarrow P_2 = \text{constant}$$

whereas $\dot{P}_1 \neq 0 \Rightarrow P_1 \neq$ constant

Here Hamiltonian is independent of time.

explicitly, so, energy is conserved.

Hence, we can say momentum associated with q_2 and energy isconserved quantity.

Notes